In the Klein case, where you project onto the hyperboloid from $(0, 0, 0)$ through the disk at $t = 1$, a straight line in the disk projects into a plane intersecting the hyperboloid.

He says that the planes intersecting the hyperboloid are geodesics in the hyperboloid, so if you believe that, that answers that part of the question.

I figured I'd get my hands dirty with connection coefficients and metrics and prove that an intersection of a plane through the origin with the hyperboloid is a geodesic, just for practice

The projection from $(-1, 0, 0)$ to the hyperboloid through the circle at $t = 0$ is conformal by a "signature flip" argument. The hyperboloid corresponds to a sphere centered at $(0, 0, 0)$ by signature flip - solutions of the equation $g_{ab}x^a x^b = 1$. So if you flip the signature, the projection of the hyperboloid to the plane at $t = 0$ turns into the projection of the Riemann sphere, a unit sphere centered at the origin, onto the plane (see sec. 8.3). The projection of the Riemann sphere is a conformal projection. So if you assume the signature flip is conformal, the projection from $(-1, 0, 0)$ onto the hyperboloid is conformal. It would be interesting if the signature flip map preserved the metric on the hyperboloid (which would imply it's conformal). But it doesn't.

What is the signature flip map, exactly? It sends the hyperboloid (both the past and future parts) and the light cone onto the Riemann sphere. You can define this map by use of the projection through $(-1, 0, 0)$. This projection also works for the past part of the hyperboloid, sending the past part onto the bottom half of the Riemann sphere. Lines through the origin on the light cone go to points on the unit circle at $t = 0$ by this projection. It wouldn't be hard to prove this is a conformal map. The map from the hyperboloid to the Riemann sphere sends (t, x, y) to $(1/t, x/t, y/t)$.

If you compose the projection through the origin onto the hyperboloid, which sends lines through the $t = 1$ unit disk to the hyperboloid, with the conformal map from the hyperboloid to the Riemann sphere, you get the projection of these lines vertically onto the unit sphere. Lines projected vertically onto the unit sphere go to circles on the unit sphere, which intersect the equator at right angles. So, geodesics on the hyperboloid intersect the unit circle at right angles when they're projected on the disk at $t = 0$.

Note that geodesics on the hyperboloid are <u>not</u> sent to great circles on the Riemann sphere! If the signature-flip map preserved the metric, geodesics would be sent to geodesics. So it doesn't preserve the metric.

The signature-flip map, sending (t, x, y) to $(1/t, x/t, y/t)$ can actually be extended to all of Minkowski space except the $t = 0$ plane, although it doesn't turn Minkowski space into Euclidean space. Sending t to it does that, but I don't know how find out things about the map from the hyperboloid to the Riemann sphere using the map from t to it.

The map from the hyperboloid to the Riemann sphere gives you a hyperbolic metric on the complex plane with the unit circle taken out, which is conformal to the usual metric. I wonder if there's some analytic function on the complex plane to some Euclidean complex space, such that $|f(z) + f(w)|$ is the hyperbolic distance between z and w.

What the hyperboloid is, is possible velocity 4-vectors at the origin. So I wondered what a geodesic on the hyperboloid means? It's a kind of acceleration, moving from one 4-vector to another. But what kind of acceleration? If you choose a nice parameterization of this geodesic, it is very simple. It's constant acceleration in the direction of motion, as measured in the rest frame of what's being accelerated. The equation for the position of a particle with this acceleration is at the end of this link.

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