Q 19.16: Show that the acceleration of volume starting from rest is
$D^2(\delta V) = R_{ab}t^a t^b \delta V$
where δV is an increment of volume, and D is the rate of change wrt the observer's proper time i.e. $D = t^a\nabla_a$, t^a is the vector pointing in the observer's proper time direction.

A: I did this first in an explicit way, which helps to see what's going on. Then, I thought for a long, long time, and I finally came up with a way to use Lie derivatives, which is after the explicit derivation and shows nicely what Lie derivatives are about. But first the explicit way:

Suppose the observer is at the origin, at the center of a little rectangular solid with orthogonal side vectors M, L and N in the x, y and z directions, and γ is the geodesic of the observer's world-line.

The worldline of a point on the boundary of the rectangular solid is a geodesic γ', and after a small time τ, γ' is no longer pointing exactly in the time direction as defined by the observer - that is, the points on the boundary have a small increment of velocity.

The point $(\tau, M/2, 0, 0)$ is on the boundary of the rectangular solid. After time τ the deviation of its geodesic from the observer's geodesic is $(M/2)\tau {R_{010}}^d$. So its velocity 4-vector is $(M/2)\tau {R_{010}}^d + (1, 0, 0, 0)$, and its acceleration is $(M/2){R_{010}}^d$.

To get the volume acceleration, it's necessary to integrate the accelerations of individual points over the surface of the rectangular solid. But this is easy. Only the component of acceleration normal to the surface matters, so to find the contributions to the volume acceleration from the faces $\perp$ to M, it's only necessary to integrate $F^b {R_{0b0}}^1$ over the faces $\perp$ to M, where F is a position vector of a point on the face.

Since $((0, M/2, 0, 0) + v)^b {R_{0b0}}^1 + ((0, M/2, 0, 0) - v)^b {R_{0b0}}^1 = M{R_{010}}^1$, the integral over the faces $\perp$ to M is just $M{R_{010}}^1 LN$. Adding the integrals over all the faces, $D^2(\delta V) = {R_{0b0}}^b MLN = {R_{0b0}}^b \delta V$. In general coordinates, that's $t^a t^c {R_{abc}}^b \delta V$.

How would you apply Lie derivatives to solve this?

If you have a 3-volume element $x \wedge y \wedge z$ where x, y, z are the 1-forms in the x, y and z directions, then the volume acceleration is the Lie derivative of the 3-volume element wrt the acceleration vector $a^c = w^b {R_{0b0}}^c$.

$$\mathcal{L}_a(x \wedge y \wedge z) = \mathcal{L}_a(x) \wedge y \wedge z + x \wedge \mathcal{L}_a(y) \wedge z + x \wedge y \wedge \mathcal{L}_a(z)$$

$$\mathcal{L}_a(x)_b = a^c\nabla_c x_b + x_c\nabla_b a^c,$$

$a^c\nabla_c x_b = 0$ since $a^c = 0$ at the origin. $a^c = w^b {R_{0b0}}^c$, so $\nabla_b a^c = {R_{0b0}}^c$, so $x_c\nabla_b a^c = {R_{0b0}}^1$.

So, $\mathcal{L}_a(x) \wedge y \wedge z = {R_{0b0}}^1 \wedge y \wedge z = {R_{010}}^1 x \wedge y \wedge z$.

Summing over x, y and z, $\mathcal{L}_a(x \wedge y \wedge z) = R_{00} x \wedge y \wedge z$.

It's a nice illustration of how to use Lie derivatives. I spent days thinking about it because I didn't understand Lie derivatives and I wanted to find out how you'd use them.

There might be a really profound way to use Lie derivatives to come up with the ${R_{0b0}}^d$ parts of the Riemann tensor - by, say, looking at the time geodesics passing through a $t = 0$ surface in spacetime and finding geodesic deviation using Lie derivatives. But I haven't come up with it (yet).

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