Q 19.18: from the Bianchi identity

$$\nabla_a R_{bcd}{}^e + \nabla_b R_{cad}{}^e + \nabla_c R_{abd}{}^e = 0$$

,

derive

$$\nabla^a (2R_{ab} - Rg_{ab}) = 0$$

.

A: Rewrite the Bianchi identity as

$$\nabla_a R_{bcde} g^{ef} + \nabla_b R_{cade} g^{ef} + \nabla_c R_{abde} g^{ef} = 0$$

Contract f with a, and it turns into

$$\nabla^e R_{edcb} + \nabla_b R_{deca} g^{ea} + \nabla_c R_{deab} g^{ea} = 0$$

raising b,

$$\nabla^e R_{edc}{}^b + \nabla^b R_{dec}{}^e + \nabla_c R_{de}{}^{eb} = 0$$

contract d with b, so

$$\nabla^e R_{ebc}{}^b + \nabla^b R_{bec}{}^e - \nabla_c R_{eb}{}^{eb} = 0$$

renaming some indices and writing it in terms of the Ricci tensor,

$$2\nabla^a R_{ab} - \nabla_b R_c{}^c = 0, so$$

$$\nabla^a (2R_{ab} - Rg_{ab}) = 0!$$

Laura