

Q 19.18: from the Bianchi identity

$$\nabla_a R_{bcd}{}^e + \nabla_b R_{cad}{}^e + \nabla_c R_{abd}{}^e = 0$$

,
derive

$$\nabla^a(2R_{ab} - Rg_{ab}) = 0$$

A: Rewrite the Bianchi identity as

$$\nabla_a R_{bcde}g^{ef} + \nabla_b R_{cade}g^{ef} + \nabla_c R_{abde}g^{ef} = 0$$

Contract f with a, and it turns into

$$\nabla^e R_{edcb} + \nabla_b R_{deca}g^{ea} + \nabla_c R_{deab}g^{ea} = 0$$

raising b,

$$\nabla^e R_{edc}{}^b + \nabla^b R_{dec}{}^e + \nabla_c R_{de}{}^{eb} = 0$$

contract d with b, so

$$\nabla^e R_{ebc}{}^b + \nabla^b R_{bec}{}^e - \nabla_c R_{eb}{}^{eb} = 0$$

renaming some indices and writing it in terms of the Ricci tensor,

$$2\nabla^a R_{ab} - \nabla_b R_c{}^c = 0, \text{ so}$$

$$\nabla^a(2R_{ab} - Rg_{ab}) = 0!$$

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