The "trick" here is $0 =< \psi|(Q^* - \bar{\lambda}I)(Q - \lambda I)|\psi >=< \psi|(Q - \lambda I)(Q^* - \bar{\lambda}I)|\psi >$, since Q commutes with Q^*.

$< \psi|(Q - \lambda I)(Q^* - \bar{\lambda}I)|\psi >=< (Q^* - \bar{\lambda}I)\psi|(Q^* - \bar{\lambda}I)|\psi >= 0$.

So $(Q^* - \bar{\lambda}I)\psi = 0$.