In sec. 25.8, he says "recall the dual ${}^*F$ of the Maxwell tensor F. We could imagine a 'dual' U(1) gauge connection that has ${}^*F$ as its bundle curvature" and then he says there's a problem with using the dual curvature of a nonabelian gauge. First, though, what happens if you try to dualize the abelian electromagnetic gauge?

If you have the Maxwell tensor F_{ab}, you can recover a vector potential A from it:

$$A_b(\vec{x}) = \int_0^1 uF_{ab}(u\vec{x})x^a du$$

Similarly, if you take the Hodge dual ${}^*F_{ab}$ of F_{ab}, you can define a gauge potential from it using a potential Z derived from ${}^*F_{ab}$

$$Z_b(\vec{x}) = \int_0^1 u^*F_{ab}(u\vec{x})x^a du$$

This only works if $d^*F_{ab} = 0$, i.e. the charge-current vector $J = 0$. It's an application of the Poincare' lemma, which says that in a contractible (small, topologically simple) region, a form F with $dF = 0$ is the exterior derivative of another form.

I totally wracked my brains about it and I couldn't see how you could use ${}^*F$ as a gauge curvature *unless* $d^*F = 0$.

And after I thought about it some <u>more</u> I figured that in the application of a dual gauge connection, the field probably *would* be source-free, because the gauge connection's applied to quantum wavefunctions and when you're at the quantum level, you wouldn't have a charge-current vector. Any charges and currents would be explicit as particle wavefunctions, not as the field.

You could add any gradient $d\phi$ to Z_b: $Z'_b = Z_b + \partial\phi/\partial x^b$ gives the same ${}^*F_{ab}$.

From Z you can define a covariant derivative $\nabla_a\psi = \partial\psi/\partial x^a - ieZ_a\psi$. I guess this connection would be applied to wavefunctions.

If you have a nonabelian gauge group SU(3), then you'd have a gauge connection

$$\nabla_a\psi = \partial\psi/\partial x^a - C_a\psi$$

. Here the C_a's are matrices in the Lie group algebra of SU(3), operating on a wavefunction that has a color index. So $\psi = y_1|red> + y_2|green> + y_3|blue>$ and $|y_1|^2 + |y_2|^2 + |y_3|^2 = 1$, so that the gauge group SU(3) is acting as transformations on S^6. The dimension of the unitary group U(3) is $3^2 = 9$ (see sec. 13.10), so the dimension of the Lie algebra of SU(3), the unitary matrices of determinant 1, is 8. I read later that there are basis elements for the Lie algebra, trace-free 3×3 Hermitian matrices called Gell-Mann matrices, for the inventor of the color theory.

The C_a's are $i\times$ a Hermitian matrix. Since e^{iH} is unitary if H is Hermitian, this gives you a unitary transform if you're integrating ∇_a; taking a path integral, with the Lie algebra elements varying over space should (though I haven't shown it rigorously) integrate to a matrix in SU(3).

The gauge transformation has to be unitary because it should preserve the inner product $< \psi|\phi >$ of two wavefunctions. And the gauge transformation should not change the wavefunction of a 3-quark combination that's been antisymmetrized with respect to color, because such a particle is a free particle, so the covariant gauge derivative shouldn't affect it. That implies it has determinant 1.

The curvature of the connection $\nabla_a\psi = \partial\psi/\partial x^a - C_a\psi$ is

$$\nabla_a\nabla_b - \nabla_b\nabla_a = \frac{\partial C_a}{\partial x^b} - \frac{\partial C_b}{\partial x^a} + C_aC_b - C_bC_a$$

This is a 2-form S_{ab} with hidden color indices. With all the indices explicit, you get

$$S_{abc}{}^d\psi_d = (\frac{\partial C_{ac}{}^d}{\partial x^b} - \frac{\partial C_{bc}{}^d}{\partial x^a} + C_{ac}{}^eC_{be}{}^d - C_{bc}{}^eC_{ae}{}^d)\psi_c.$$

It's like the Riemann tensor except that the c and d indices are color indices, not space indices.

S_{ab} satisfies a Bianchi identity $\partial S_{[ab}/\partial x^{c]} = 0$, I checked.

I tried to find a curvature tensor for a connection with *both* a spacetime curvature and curvature on the color indices (the gauge curvature), but it didn't work, that is the commutator $(\nabla_a\nabla_b - \nabla_b\nabla_a)\psi$ didn't work out to something multiplied by just ψ. Trying to quantize gravity!

You can find the Hodge dual $^*S_{ab}$ and try to interpret it as a curvature tensor. But, with a nonabelian gauge the commutator $C_aC_b - C_bC_a$ doesn't disappear, so the gauge curvature doesn't look like the exterior derivative of a form. So the Poincare' lemma might not apply. If you could show that $^*S_{ab}$ doesn't satisfy the Bianchi identity $\partial^*S_{[ab}/\partial x^{c]} = 0$, that would show that $^*S_{ab}$ isn't a curvature tensor, at least for a connection of the form $\nabla_a\psi = \partial\psi/\partial x^a - C_a\psi$ - since I checked that S_{ab} does satisfy this Bianchi identity! The terms in the Bianchi identity for $^*S_{ab}$ are a lot of complicated stuff that doesn't look like it would have a habit of summing to 0.

If $^*S_{ab}$ *did* satisfy the Bianchi identity $\partial^*S_{[ab}/\partial x^{c]} = 0$, maybe that would mean it's a curvature tensor for a connection of the form $\nabla_a\psi = \partial\psi/\partial x^a - C_a\psi$. I don't know, since the Poincare' lemma doesn't necessarily apply.

So that is my best take on a confusing exercise!