27.16 If a connected 3-space is isotropic around 2 separate points p and q, why is it homogeneous?

A: It isn't true! Counter-example: if the space is the 3-sphere - i.e. the set of points (x, y, z, w), with $x^2+y^2+z^2+w^2 = 1$. Let p be a point on the sphere. You could have a matter distribution that was a function only of distance from p, and it'd be isotropic around p and $-p$ too.

This counterexample works because any distance-preserving rotation that fixes p also fixes $-p$.

Suppose this isn't true for the 2 separate points p and q; for example, you can rotate around q a little bit and it'll move p a little. Then the space is homogeneous.

Take the set of points z such that there's no isometry (distance preserving transform) sending z to p. Let d be the greatest lower bound of the distances from the points z to p.

You can nudge p a little bit by rotating around q: suppose you can move p to $p + \Delta p$ by rotating around q. By continuously varying the rotation around q from no rotation to the rotation that moves p to $p+\Delta p$, you can find a path from p to $p+\Delta p$, such that p can be moved to any point on the path. Choose a point p' on the path that is closer than d to p.

If there's no isometry sending z to p, then if you rotate z around p to z', there's also no isometry sending z' to p (if there were, you could first send z to z' and then z' to p).

So, if you travel on the geodesic from p to p', there's a point z' at the same distance as z from p, such that there's no isometry sending z' to p. z' can be found that's closer than d to p'. Transforming p' back to p by a rotation around q transforms z' to a point z'' that's closer than d to p. So there's an isometry that sends p to z''. Composing that with the rotation around q that sends p to p', there's an isometry that sends p to z', a contradiction.

If $d = 0$, so there are points z arbitrarily close to p with no isometry sending p to z, there must be some such z on the path from p to $p + \Delta p$, which is a contradiction.

Now for things beyond just doing the exercise! This uses some Riemannian geometry. Suppose the space is a complete Riemannian 3-manifold, and it has a lot of symmetry: there's an isometry group $G(p)$ which fixes a point p and is transitive on the unit tangent vectors at p, and fixes another point q, and similarly a isometry group $G(q)$, transitive on the unit tangent vectors at q, which fixes q and p.

There's a geodesic from p to q because the space is complete. Since $G(p)$ is transitive on the geodesics through p, any geodesic through p has to go through q. And any geodesic from q goes through p.

Let q be the closest point to p other than p that's fixed by $G(p)$. There is a closest such point because there is an open neighborhood V of p inside of which the exponential map at p is a diffeomorphism. And the set of fixed points of $G(p)$ is closed because the isometries in $G(p)$ are continuous.

Because the space is complete, any point x on the manifold is connected to p by a geodesic. So x is on one of the geodesics joining p and q, and the space is compact.

Also there can't be any other fixed points of $G(p)$ other than p or q, because any other fixed point would be on a geodesic between p and q, and I chose q to be the closest one.

Can there be points other than q in the tangent space at p where the exponential map is singular? They would be singularities of order 1, which means that the kernel of the jacobian of the exponential map is dimension 1. There would be a spherical shell of singularities in the tangent space at p. But an isometry in the tangent space is a rotation of the spherical shell, and it has to have a fixed point. So the singularity at that point has to have rotational symmetry around that point - so it can't be of dimension 1.

I think this means the space has to be homeomorphic to the 3-sphere, and the 3-sphere is the only counterexample to this exercise.

I actually showed that if the space is a complete Riemannian manifold and you just assume that there's an isometry group $G(p)$ transitive on the unit tangent vectors at p, fixing another point q, then all the geodesics through q have to go through p - so that the same situation happens, the space is composed entirely of geodesics between p and q and it has to be homeomorphic to S^3.

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