28.4 Find coordinates within de Sitter space so the metric is $ds^2 = d\tau^2 - e^{A\tau}(dx^2 + dy^2 + dz^2)$

A: God, but this was hard to figure out! He drops a lot of subtle hints, but I don't always get it until after a lot of puzzling.

de Sitter space is the hyperboloid $x^2 + y^2 + z^2 + w^2 - t^2 = A^2$ in Minkowski 5-space.

If you take $x, y, z = 0$, you get a curve $w + t = A^2/(w - t)$ on the hyperboloid. So $d(t + w) = A^2 d(t - w)/(t - w)^2$. The metric in Minkowski 5-space is $ds^2 = dt^2 - dw^2 - dx^2 - dy^2 - dz^2$, and on this curve it is $ds^2 = dt^2 - dw^2 = d(t - w)d(t + w) = A^2 d(t - w)^2/(t - w)^2 = d\tau^2$. So $d\tau = Ad(t - w)/(t - w)$, so try $e^{\tau/A} = t - w$.

Since he says that $t - w$ is the time coordinate, τ is a function just of $t - w$. So we take $e^{\tau/A} = t - w$ on the entire hyperboloid, and see what happens to the metric.

$t + w = e^{-\tau/A}(x^2 + y^2 + z^2 - A^2)$ on the hyperboloid so
$d(t + w) = e^{-\tau/A}(-1/A(x^2 + y^2 + z^2 - A^2)d\tau + 2xdx + 2ydy + 2zdz)$.

So the metric is

$$d(t-w)d(t+w)-(dx^2+dy^2+dz^2) = d\tau^2 - \frac{(x^2 + y^2 + z^2)}{A^2}d\tau^2 + \frac{(2xdx + 2ydy + 2zdz)}{A}d\tau - (dx^2+dy^2+dz^2).$$

If you substitute $x_1 = e^{-\tau/A}x$ and similarly for y and z, this becomes
$ds^2 = d\tau^2 - e^{2\tau/A}(dx_1^2 + dy_1^2 + dz_1^2)$.

I didn't guess quite right what A would be, but it's the right form.

In a constant-τ slice, this is flat Euclidean 3-space. A constant-τ slice through the hyperboloid is an intersection with a plane $t - w = c$, c is a constant. The equation of the slice is $x^2 + y^2 + z^2 = A^2 + c(t + w)$. For a given $t + w$, it's a 2-D sphere! So $t + w$ is sort of the radius of a 2-sphere, and you build up a 3-space with concentric 2-spheres! Since $t - w = c, dt = dw$ on the slice, so $ds^2 = dx^2 + dy^2 + dz^2$ - just the Euclidean metric. The 2-D spheres don't care that the t, w coordinates vary with their radius, because this doesn't appear in the metric. So it looks like flat 3-space metrically! This flatness carries over when you convert to x_1, y_1, z_1 coords.

Laura