28.6 Why is the Ricci tensor proportional to the metric in de Sitter and anti de Sitter space?

A: I really puzzled over this! But I think it's because the metric gives an exponential volume expansion over proper time. For both de Sitter and anti de Sitter space. For de Sitter space, the metric is $ds^2 = d\tau^2 - e^{2\tau}(dx^2 + dy^2 + dz^2)$.

For anti de Sitter space, interestingly, you can find a metric using the same technique as for de Sitter space, as I posted in exercise 28.4. In the same way you take constant-$\tau$ lines to be the intersections of constant $t - x$ planes with the surface. But with anti de Sitter space, the metric you get is $ds^2 = -d\tau^2 + e^{2\tau}(dw^2 - dy^2 - dz^2)$. So just like de Sitter space, it's a spacetime, one timelike dimension, 3 spacelike dimensions, only $\tau$ is a spacelike dimension!

The volume is expanding exponentially with $\tau$ in both de Sitter and anti de Sitter space. In section 19.6, it says the Ricci tensor has info about the volume acceleration: $D^2(\delta V) = R_{ab}t^a t^b \delta V$. This is true actually for geodesics starting out parallel from $\delta V$ in any direction, not just in a timelike direction: $\nabla_u \nabla_u \delta V = R_{ab}u^a u^b \delta V$, although if $u$ isn't timelike, $\delta V$ wouldn't be exactly a volume element, it'd be the "volume" of an element of the 3-space orthogonal to $u$.

From the metric for de Sitter and anti de Sitter space, we already know what the volume acceleration is for $\delta V$ in the $\tau$ direction, because lines of constant $x, y, z$ and varying $\tau$ are geodesics. So $V(\tau) = e^{3\tau}V_{\tau=0}$ so $D^2(\delta V) = 9e^{3\tau}V_{\tau=0}$. So $R_{00} = 9$ (I actually couldn't get it to come out exactly right when I calculated the Ricci tensor).

If you know $R_{00}$ you can get the other components $R_{ab}$. Suppose $t^a$ is another timelike geodesic, and suppose there's an isometry mapping the $\tau$ geodesic starting from a point, to $t^a$. The Ricci tensor looks the same in the new coordinates; in those coordinates $R_{ab}t^a t^b = R_{00}$ is 9 again. Since $R_{ab}t^a t^b$ is a scalar, it's an invariant, so $R_{ab}t^a t^b = 9$ in the original coordinates also. That's assuming there *is* an isometry taking the $\tau$ geodesic to any other timelike geodesic. So, say in de Sitter space, if you transform the vector $t = (\tau, x, y, z) = (1, 0, 0, 0)$ to $t^a = (c, d, 0, 0)$, normalized by dividing by $\sqrt{c^2 - e^{2\tau}d^2}$, then $R_{ab}t^a t^b = R_{00} = 9$.

So this determines the other components of $R_{ab}$ and it means that $R_{ab}$ is a multiple of $g_{ab}$; $R_{ab}t^a t^b = (9c^2 + 2R_{01}cd + R_{11}d^2)/(c^2 - e^{2\tau}d^2) = 9$. Since this holds for general $c, d, R_{01} = 0$ and $R_{11} = -9e^{2\tau}$. And so on.

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