29.13

You get the density matrix for a spin-1 state $|a>|b> + |b>|a>$ by measuring the state relative to orthogonal states $<c|$ and $<d|$, $<c|d>= 0$. The density matrix is
$(<c|a>|b> + <c|b>|a>)(<a|c><b| + <b|c><a|) +$
$(<d|a>|b> + <d|b>|a>)(<a|d><b| + <b|d><a|)$. It needs normalizing, but that's the general idea.

In the example in the text, the density matrix $L = \frac{1}{3}|\leftarrow><\leftarrow| + \frac{2}{3}|\uparrow><\uparrow|$. Here, the probability that the person on Titan sees the $<c| = <\rightarrow|$ state is
$(<a|c><b| + <b|c><a|)(<c|a>|b> + <c|b>|a>) = \frac{1}{3}$. If the Titanian sees the $<c|$ state, the state on Earth becomes $<c|a>|b> + <c|b>|a>$ (which needs normalizing).

The density matrix doesn't depend on the particular states $|c>$ and $|d>$, as long as they're orthogonal. I checked this by calculating. But you can see this without calculating; as he says in the spin-0 case, a Titanian's decision about which direction to measure one particle of an entangled EPR pair shouldn't affect the probabilities on Earth. Otherwise there'd be a way to send signals from Titan to Earth, faster than light. This probably means the density matrix can't be affected by the direction of measurement on Titan, since for any projector E, the probability of a YES measurement = trace(ED). So the trace of ED isn't affected by the direction of measurement, for any E.

If you do exercise 29.12, you see that the eigenvectors of the density matrix for the state $|a>|b> + |b>|a>$ are $|a> + |b>$ and $|a> - |b>$, if $<a|a> = <b|b> = 1$.

This gives a hint about how to obtain an arbitrary density matrix. The eigenvectors of the density matrix are orthogonal because it's Hermitian. So, let's rotate the $|a>$ and $|b>$ vectors so $|a> + b>$ is in the $|\uparrow>$ direction and $|a> - |b>$ is in the $|\downarrow>$ direction. Then the density matrix in the $|\uparrow>, |\downarrow>$ basis will be diagonal, and all you need to do to show that you can get an arbitrary density matrix is to show you can get the eigenvalues of an arbitrary density matrix.

Suppose then that $|a> = \alpha|\uparrow> + \beta|\downarrow>$ and $|b> = \alpha|\uparrow> - \beta|\downarrow>$ where $|\alpha|^2 + |\beta|^2 = 1$. Since the density matrix is independent of the c/d direction, let's take $<c| = <\uparrow|$, $<d| = <\downarrow|$. Just to make things giddily easy.

Here

$$|a>|b> + |b>|a> = 2\alpha^2|\uparrow> - 2\beta^2|\downarrow> .$$

After normalizing this becomes

$$\frac{\alpha^2|\uparrow> - \beta^2|\downarrow>}{\sqrt{|\alpha|^4 + |\beta|^4}}.$$

The probability of an $<\uparrow|$ measurement is $\frac{|\alpha|^4}{|\alpha|^4+|\beta|^4}$ and of a $<\downarrow|$ measurement is $\frac{|\beta|^4}{|\alpha|^4+|\beta|^4}$.

So the density matrix is

$$\begin{pmatrix} \frac{|\alpha|^4}{|\alpha|^4+|\beta|^4} & 0 \\ 0 & \frac{|\beta|^4}{|\alpha|^4+|\beta|^4} \end{pmatrix}$$

Since a density matrix has eigenvalues that sum to 1 and are between -1 and 1, any density matrix eigenvalues can be obtained this way. That's it!!! (loud whistle, exit into 3rd dimension).