Q: Find a formula for gravitational time dilation near a body mass \( M \), given a timelike Killing vector \( \kappa \).

A: Suppose you have a particle of mass \( m \) falling from rest at \( \infty \), straight towards \( M \). It’s following a geodesic so the product \( p_a \kappa^a \) is conserved (see exercise 19.14). At \( \infty \), \( \kappa = (1, 0, 0, 0) \) and \( p_a = (mc^2, 0, 0, 0) \). At a distance \( R \) from \( M \), the particle has kinetic energy \( mMG/R \). So its energy is now \( m(c^2 + MG/R) \). Since the spacetime is stationary, the space components of \( \kappa \) are 0, \( \kappa = (\kappa^0, 0, 0, 0) \). By conservation of \( p_a \kappa^a \), \( mc^2 = \kappa^0 (m(c^2 + MG/R)) \), so \( \kappa^0 = 1/(1 + MG/Rc^2) \).

So why is this time dilation? It is in this sense at least: \( p_0 = \hbar \omega \) for a light ray, where \( \omega \) is frequency. Coming out of the gravitational field, by the conservation of \( p_a \kappa^a \), the frequency \( \omega \) decreases by a factor of \( 1 + MG/Rc^2 \). That means that time seems to be going more slowly at the source of the light.

Also, \( \kappa \) is a vector field that describes how a spacelike (constant-time) slice of the spacetime evolves in time. You would evolve the spacelike slice an infinitesimal amount in time, by applying the \( \kappa \) vector to move each point a little along a geodesic in the time direction. Since the \( \kappa \) vector is shorter closer to \( M \), that means that time moves more slowly closer to \( M \).