There are 2 mass distributions ρ_1 and ρ_2, and ρ_2 is rigidly translated in space from ρ_1, so $\rho_2(\vec{x}+\vec{x'}) = \rho_1(\vec{x})$. Also the new position of the mass distribution is at the same energy with respect to any external gravitational field, so the potentials ϕ_1 and ϕ_2 are also rigidly translated in space: $\phi_2(\vec{x}+\vec{x'}) = \phi_1(\vec{x})$.

The gravitational self-energy of the difference between the two mass distributions is the integral over all space (how grand it feels to say such things!)

$$-\int (\rho_2(\vec{x}) - \rho_1(\vec{x}))(\phi_2(\vec{x}) - \phi_1(\vec{x}))dx^3$$

The exercise asks you to show that this integral is equal to $\int (\rho_2(\vec{x}) - \rho_1(\vec{x}))\phi_1(\vec{x})dx^3$, the change in energy to move the mass distribution to the 2nd position, leaving the potential energy it generates in the 1st position!

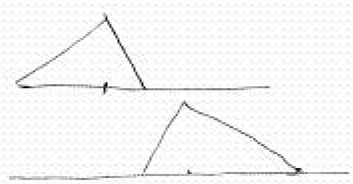
You might think that this is real easy. It's obvious that $\int \rho_1(\vec{x})\phi_1(\vec{x})dx^3 = \int \rho_2(\vec{x})\phi_2(\vec{x})dx^3$. And it might look obvious that $\int \rho_1(\vec{x})\phi_2(\vec{x})dx^3 = \int \rho_2(\vec{x})\phi_1(\vec{x})dx^3$.

You just switch positions 1 and 2, right?

Not really.

Suppose ρ_1 looks like

and ϕ_2 looks like



Then $\int \rho_1(\vec{x})\phi_2(\vec{x})dx^3 = 0$, but $\int \rho_2(\vec{x})\phi_1(\vec{x})dx^3 \neq 0$.

What one needs is a way to relate ρ and ϕ. They *are* related, by Poisson's eq. $\nabla^2\phi = -4\pi\rho$.

That suggested to me Fourier transforming ρ and ϕ, since if $\phi = e^{ixp}$, ϕ is proportional to ρ, so $\int \rho_1(\vec{x})\phi_2(\vec{x})dx^3$ definitely is equal to $\int \rho_2(\vec{x})\phi_1(\vec{x})dx^3$.

The Fourier transform of $\phi_1(\vec{x})$ is $\int_{-\infty}^{\infty} g(\vec{p})e^{i\vec{x}\cdot\vec{p}}dp$ (dropping a pesky factor of $\sqrt{2\pi}$). $\overline{g(\vec{p})} = g(-\vec{p})$, so that ϕ_1 is real.

So the Fourier transform of ρ_1 is $\int_{-\infty}^{\infty} |q|^2 g(\vec{q})e^{i\vec{x}\cdot\vec{q}}dq$ (dropping another pesky factor of 4π)

What we'd like to show is that

$$\int g(\vec{p})e^{i\vec{x}\cdot\vec{p}}|q|^2 g(\vec{q})e^{i\vec{x}\cdot\vec{q}}e^{i\vec{x'}\cdot\vec{q}}dpdqdx = \int g(\vec{p})e^{i\vec{x}\cdot\vec{p}}e^{i\vec{x'}\cdot\vec{p}}|q|^2 g(\vec{q})e^{i\vec{x}\cdot\vec{q}}dpdqdx$$

where the integral is from $-\infty$ to ∞ in p, q and x.

Switching the order of integration to integrate over x first, we notice that $\int e^{i\vec{x}\cdot(\vec{p}+\vec{q})}dx = 0$ unless $\vec{q} = -\vec{p}$, and in that case it's a delta function.

So the equality we want to prove simplifies to

$$\int g(\vec{p})|p|^2 g(-\vec{p}) e^{-i\vec{x'}\cdot\vec{p}} dp = \int g(\vec{p})|p|^2 g(-\vec{p}) e^{i\vec{x'}\cdot\vec{p}} dp$$

Since $\overline{g(\vec{p})} = g(-\vec{p})$ this becomes

$$\int |g(\vec{p})|^2 |p|^2 e^{-i\vec{x'}\cdot\vec{p}} dp = \int |g(\vec{p})|^2 |p|^2 e^{i\vec{x'}\cdot\vec{p}} dp$$

and that is true because both of the integrals are real-valued - since they are the integrals $\int \rho_1 \phi_2 dx$ and $\int \rho_2 \phi_1 dx$.

The gravitational self-energy would actually be twice the change in energy to move the mass distribution away from its potential, but Roger Penrose doesn't apparently care about scale factors here.

That shows it if the mass distributions aren't being moved to a different potential energy in an external gravitational field. What if they are? Then $\phi_2(\vec{x} + \vec{x'}) = \phi_1(\vec{x}) + mgh$, and the integral $-\int (\rho_2(\vec{x}) - \rho_1(\vec{x}))(\phi_2(\vec{x}) - \phi_1(\vec{x}))dx^3$ is changed from the case without the mgh term by $-\int mgh(\rho_2(\vec{x}) - \rho_1(\vec{x}))dx^3$, which is 0, so still the gravitational self-energy is the same as the energy required to displace the mass distribution to its new position, leaving the potential it generated in the old position (except for a factor of 2).

If the change in potential isn't uniform over space, this equality would no longer hold.

Similarly, if you compress the mass while moving it, the gravitational self-energy would have an additional term from the compression in position 2, and this equality wouldn't hold. I didn't work out the full gory details of it, but the very goriality convinced me there was no need to. The integrals were not going to turn out equal without the help of an erasing demon.

Laura