Exercise 33.11 If a twistor (Z^0, Z^1, Z^2, Z^3) has norm 0, show that it has a corresponding event (t, x, y, z) in spacetime, meaning

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix}.$$

If the norm of a twistor (Z^0, Z^1, Z^2, Z^3) is 0, $\overline{Z^2}Z^0 + \overline{Z^3}Z^1 = (Z^2, Z^3) \cdot (Z^0, Z^1)$ is pure imaginary.

Since

$$\begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} = tI + \begin{pmatrix} z & x+iy \\ x-iy & -z \end{pmatrix},$$

t can be found by finding the component of (Z^0, Z^1) in the (Z^2, Z^3) direction.

$$\text{Since } (Z^{0\prime}, Z^{1\prime}) = (Z^0, Z^1) - \frac{(\overline{Z^2}Z^0 + \overline{Z^3}Z^1)}{|Z^2|^2 + |Z^3|^2}(Z^2, Z^3)$$

satisfies $\overline{Z^{0\prime}}Z^2 + \overline{Z^{1\prime}}Z^3 = 0$, t can be taken to be

$$t = -\sqrt{2}i\frac{(\overline{Z^2}Z^0 + \overline{Z^3}Z^1)}{|Z^2|^2 + |Z^3|^2},$$

which is real.

So, assume that $t = 0$ and $\overline{Z^0}Z^2 + \overline{Z^1}Z^3 = 0$.

$$\text{The matrix} \begin{pmatrix} z & x+iy \\ x-iy & -z \end{pmatrix}$$

has eigenvalues $r = \sqrt{x^2 + y^2 + z^2}$ and $-r$. Since

$$\frac{|Z^0|^2 + |Z^1|^2}{|Z^2|^2 + |Z^3|^2} = \frac{r^2}{2},$$

r is determined, and r can be taken to be 1 by scaling (Z^0, Z^1) so that $(Z^{0\prime}, Z^{1\prime}) = \frac{1}{r}(Z^0, Z^1)$.

$\begin{pmatrix} z & x+iy \\ x-iy & -z \end{pmatrix}$ has eigenvectors $v_1 = (x+iy, 1-z)$ and $v_2 = (x+iy, -z-1)$, which are orthogonal since it's Hermitian. So (Z^0, Z^1) and (Z^2, Z^3) can be expressed in terms of v_1 and v_2. We want to find v_1 and v_2 such that $(Z^2, Z^3) = av_1 + bv_2$ and $(Z^0, Z^1) = \frac{i}{\sqrt{2}}(av_1 - bv_2)$, where a and b are complex numbers.

So $(-\sqrt{2}iZ^0 + Z^2, -\sqrt{2}iZ^1 + Z^3) = 2av_1 = 2a(x+iy, 1-z)$
and $(\sqrt{2}iZ^0 + Z^2, \sqrt{2}iZ^1 + Z^3) = 2bv_2 = 2b(x+iy, -1-z)$.

$$\frac{a}{b} = \frac{-\sqrt{2}iZ^0 + Z^2}{\sqrt{2}iZ^0 + Z^2}, \text{ and since } \frac{-\sqrt{2}iZ^1 + Z^3}{\sqrt{2}iZ^1 + Z^3} = \frac{a}{b}\frac{1-z}{-1-z},$$

$$\frac{1-z}{-1-z} = \frac{(-\sqrt{2}iZ^1 + Z^3)}{(\sqrt{2}iZ^1 + Z^3)}\frac{(\sqrt{2}iZ^0 + Z^2)}{(-\sqrt{2}iZ^0 + Z^2)}, \; z \text{ is determined.}$$

$\frac{1-z}{-1-z}$ is a negative real number, as you can see by multiplying the numerator and denominator

$$\text{of } \frac{(-\sqrt{2}iZ^1 + Z^3)}{(\sqrt{2}iZ^1 + Z^3)}\frac{(\sqrt{2}iZ^0 + Z^2)}{(-\sqrt{2}iZ^0 + Z^2)} \text{ by the conjugate of the denominator.}$$

So $|z| \leq 1$. Then you can find

$$a = \frac{-\sqrt{2}iZ^1 + Z^3}{2(1-z)}, \text{ and knowing } a, \text{ you can find } b, x \text{ and } y.$$

To check that it all works, find

$$\frac{(x+iy)}{(1-z)}\frac{(x-iy)}{(-1-z)} = \frac{(-\sqrt{2}iZ^0 + Z^2)}{(-\sqrt{2}iZ^1 + Z^3)}\frac{(-\sqrt{2}i\overline{Z^0} + \overline{Z^2})}{(-\sqrt{2}i\overline{Z^1} + \overline{Z^3})}, \text{ which works out to be -1, as it should.}$$

So yes, the twistor with 0 norm does have a spacetime event associated with it!

Explicit formulas for x, y, z:
If $r = 1$ and $t = 0$ then

$$x = \frac{i}{2\sqrt{2}}\frac{\left[-2(Z^0)^2 + 2(Z^1)^2 - (Z^2)^2 + (Z^3)^2\right]}{Z^0Z^3 - Z^1Z^2}$$

$$y = \frac{-1}{2\sqrt{2}}\frac{\left[2(Z^0)^2 + 2(Z^1)^2 + (Z^2)^2 + (Z^3)^2\right]}{Z^0Z^3 - Z^1Z^2}$$

$$z = \frac{i}{\sqrt{2}}\frac{(2Z^0Z^1 + Z^2Z^3)}{(Z^0Z^3 - Z^1Z^2)}$$

What if $Z^0Z^3 - Z^1Z^2 = 0$? That means that $(Z^0, Z^1) = \alpha(Z^2, Z^3)$ for some α. But t was chosen so that (Z^0, Z^1) is orthogonal to (Z^2, Z^3). So $\alpha = 0$.

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