Exercise 33.16

to show that the locus of points in $\text{CM}^#$ that are incident with a given twistor $(Z^1, Z^2, Z^3, Z^4)$ is a self-dual plane, meaning the associated 2-form is self-dual. Similarly, the points in $\text{CM}^#$ that are incident with the complex conjugate of the twistor are an anti self-dual plane.

I’ll just sketch this out, it’s too much algebra to write it all up explicitly in Latex. The metric in Minkowski space is $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$, and it’s the same in complexified Minkowski space, no complex conjugation is used.

A vector $(t, x, y, z)$ in the plane that’s incident with $(Z^1, Z^2, Z^3, Z^4)$ satisfies

$$\begin{pmatrix} t + z & x + iy \\ x - iy & t - z \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$ 

So

$$x = \frac{-(Z^2)^2(t + z) - (Z^3)^2(t - z)}{2Z^2Z^3},$$
$$y = \frac{-(Z^2)^2(t + z) + (Z^3)^2(t - z)}{2iZ^2Z^3}.$$ 

Setting $t = 0, z = 1$ and $t = 1, z = 0$, to get basis vectors for the plane:

$$\bar{v}^a = \begin{pmatrix} 0, \frac{(Z^3)^2 - (Z^2)^2}{2Z^2Z^3}, \frac{-(Z^2)^2 - (Z^3)^2}{2iZ^2Z^3}, 1 \end{pmatrix}$$
$$\bar{w}^a = \begin{pmatrix} 1, \frac{-(Z^2)^2 - (Z^3)^2}{2Z^2Z^3}, \frac{-(Z^2)^2 + (Z^3)^2}{2iZ^2Z^3}, 0 \end{pmatrix}$$

Lowering the index on $\bar{v}$ and $\bar{w}$ to get 1-forms,

$$\bar{v}_a = \begin{pmatrix} 0, \frac{(Z^2)^2 - (Z^3)^2}{2Z^2Z^3}, \frac{(Z^2)^2 + (Z^3)^2}{2iZ^2Z^3}, -1 \end{pmatrix}$$
$$\bar{w}_a = \begin{pmatrix} 1, \frac{(Z^2)^2 + (Z^3)^2}{2Z^2Z^3}, \frac{(Z^2)^2 - (Z^3)^2}{2iZ^2Z^3}, 0 \end{pmatrix}$$

Taking the wedge product $(v \wedge w)_{ab} = \frac{1}{2}(v_aw_b - v_bw_a)$,

$$(v \wedge w)_{tz} = \frac{-1}{2},$$
$$(v \wedge w)_{tx} = \frac{(Z^2)^2 - (Z^3)^2}{4Z^2Z^3}.$$
$$(v \wedge w)_{ty} = \frac{(Z^2)^2 + (Z^3)^2}{4iZ^2Z^3}$$

$$(v \wedge w)_{xx} = \frac{-(Z^2)^2 - (Z^3)^2}{4Z^2Z^3}$$

$$(v \wedge w)_{zy} = \frac{-(Z^2)^2 + (Z^3)^2}{4iZ^2Z^3}$$

$$(v \wedge w)_{xy} = i/2.$$

$v \wedge w$ is the 2-form tangent to the plane.

The Hodge dual $^\ast (v \wedge w)_{ab} = \frac{1}{2} \epsilon_{abcd} (v \wedge w)^{cd}$. Here $\epsilon_{txyz} = 1$. $v \wedge w$ being self-dual means that $^\ast (v \wedge w)_{ab} = i v \wedge w$.

Raising the indices on $v \wedge w$,

$$(v \wedge w)^{tz} = \frac{1}{2},$$

$$(v \wedge w)^{tx} = \frac{-(Z^2)^2 + (Z^3)^2}{4Z^2Z^3}$$

$$(v \wedge w)^{ty} = \frac{-(Z^2)^2 - (Z^3)^2}{4iZ^2Z^3}$$

$$(v \wedge w)^{xz} = \frac{-(Z^2)^2 - (Z^3)^2}{4Z^2Z^3}$$

$$(v \wedge w)^{zy} = \frac{-(Z^2)^2 + (Z^3)^2}{4iZ^2Z^3}$$

$$(v \wedge w)^{xy} = i/2.$$

and you can verify that it’s self-dual by doing the calculation.

A vector $(t, x, y, z)$ in the plane determined by the dual twistor $(Z^2, Z^3, Z^0, Z^1)$ satisfies

$$
\begin{pmatrix}
  t + z & x - iy \\
  x + iy & t - z
\end{pmatrix}
\begin{pmatrix}
  Z^2 \\
  Z^3
\end{pmatrix}
= \begin{pmatrix}
  0 \\
  0
\end{pmatrix},
$$

and if you go through the same calculation, it turns out that this plane is anti self dual, which means that $^\ast (v' \wedge w')_{ab} = -iv' \wedge w'$ for $v' \wedge w'$ tangent to the conjugate plane.

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