Exercise 33.16

to show that the locus of points in $CM^{\#}$ that are incident with a given twistor (Z^1, Z^2, Z^3, Z^4) is a self-dual plane, meaning the associated 2-form is self-dual. Similarly, the points in $CM^{\#}$ that are incident with the complex conjugate of the twistor are an anti self-dual plane.

I'll just sketch this out, it's too much algebra to write it all up explicitly in Latex. The metric in Minkowski space is $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$, and it's the same in complexified Minkowski space, no complex conjugation is used.

A vector (t, x, y, z) in the plane that's incident with (Z^1, Z^2, Z^3, Z^4) satisfies

$$\begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\text{So } x = \frac{-(Z^2)^2(t+z) - (Z^3)^2(t-z)}{2Z^2Z^3}$$

$$y = \frac{-(Z^2)^2(t+z) + (Z^3)^2(t-z)}{2iZ^2Z^3}.$$

Setting $t = 0, z = 1$ and $t = 1, z = 0$, to get basis vectors for the plane:

$$\vec{v^a} = \left(0, \frac{(Z^3)^2 - (Z^2)^2}{2Z^2Z^3}, \frac{-(Z^2)^2 - (Z^3)^2}{2iZ^2Z^3}, 1\right)$$

$$\vec{w^a} = \left(1, \frac{-(Z^2)^2 - (Z^3)^2}{2Z^2Z^3}, \frac{-(Z^2)^2 + (Z^3)^2}{2iZ^2Z^3}, 0\right)$$

Lowering the index on $\vec{v}$ and $\vec{w}$ to get 1-forms,

$$\mathbf{v_a} = \left(0, \frac{(Z^2)^2 - (Z^3)^2}{2Z^2Z^3}, \frac{(Z^2)^2 + (Z^3)^2}{2iZ^2Z^3}, -1\right)$$

$$\mathbf{w_a} = \left(1, \frac{(Z^2)^2 + (Z^3)^2}{2Z^2Z^3}, \frac{(Z^2)^2 - (Z^3)^2}{2iZ^2Z^3}, 0\right)$$

Taking the wedge product $(\mathbf{v} \wedge \mathbf{w})_{ab} = \frac{1}{2}(\mathbf{v_a w_b} - \mathbf{v_b w_a})$,

$$(\mathbf{v} \wedge \mathbf{w})_{tz} = \frac{-1}{2},$$

$$(\mathbf{v} \wedge \mathbf{w})_{tx} = \frac{(Z^2)^2 - (Z^3)^2}{4Z^2Z^3}$$

$$(\mathbf{v} \wedge \mathbf{w})_{ty} = \frac{(Z^2)^2 + (Z^3)^2}{4iZ^2 Z^3}$$

$$(\mathbf{v} \wedge \mathbf{w})_{zx} = \frac{-(Z^2)^2 - (Z^3)^2}{4Z^2 Z^3}$$

$$(\mathbf{v} \wedge \mathbf{w})_{zy} = \frac{-(Z^2)^2 + (Z^3)^2}{4iZ^2 Z^3}$$

$$(\mathbf{v} \wedge \mathbf{w})_{xy} = i/2.$$

$\mathbf{v} \wedge \mathbf{w}$ is the 2-form tangent to the plane.
The Hodge dual ${}^*(\mathbf{v} \wedge \mathbf{w})_{ab} = \frac{1}{2}\epsilon_{abcd}(v \wedge w)^{cd}$. Here $\epsilon_{txyz} = 1$.
$\mathbf{v} \wedge \mathbf{w}$ being self-dual means that ${}^*(\mathbf{v} \wedge \mathbf{w})_{ab} = i\mathbf{v} \wedge \mathbf{w}$.
Raising the indices on $\mathbf{v} \wedge \mathbf{w}$,

$$(\mathbf{v} \wedge \mathbf{w})^{tz} = \frac{1}{2},$$

$$(\mathbf{v} \wedge \mathbf{w})^{tx} = \frac{-(Z^2)^2 + (Z^3)^2}{4Z^2 Z^3}$$

$$(\mathbf{v} \wedge \mathbf{w})^{ty} = \frac{-(Z^2)^2 - (Z^3)^2}{4iZ^2 Z^3}$$

$$(\mathbf{v} \wedge \mathbf{w})^{zx} = \frac{-(Z^2)^2 - (Z^3)^2}{4Z^2 Z^3}$$

$$(\mathbf{v} \wedge \mathbf{w})^{zy} = \frac{-(Z^2)^2 + (Z^3)^2}{4iZ^2 Z^3}$$

$$(\mathbf{v} \wedge \mathbf{w})^{xy} = i/2.$$

and you can verify that it's self-dual by doing the calculation.
A vector (t, x, y, z) in the plane determined by the dual twistor $(\overline{Z^2}, \overline{Z^3}, \overline{Z^0}, \overline{Z^1})$ satisfies

$$\begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \begin{pmatrix} \overline{Z^2} \\ \overline{Z^3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and if you go through the same calculation, it turns out that this plane is anti self dual, which means that ${}^*(\mathbf{v}' \wedge \mathbf{w}')_{ab} = -i\mathbf{v}' \wedge \mathbf{w}'$ for $\mathbf{v}' \wedge \mathbf{w}'$ tangent to the conjugate plane.

Laura