Exer. 33.18

The way you make that traveling twisty thing that's depicted in Fig. 33.15 is to start with a twistor (Z^0, Z^1, Z^2, Z^3) with nonzero norm. The twistors that are orthogonal to (Z^0, Z^1, Z^2, Z^3) are a plane in $\mathbb{PT}$, because any linear combination of twistors orthogonal to (Z^0, Z^1, Z^2, Z^3) is also orthogonal to it. That's what he means by "a dual twistor determines a plane in $\mathbb{PT}$".

Out of those twistors orthogonal to (Z^0, Z^1, Z^2, Z^3), the ones with norm 0 are selected. A twistor with norm 0 defines a null geodesic (light ray) in Minkowski space, as was shown in Exer. 33.11 and Exer. 33.12. The $t = 0$ point on the light ray, and the direction of the light ray from that point, define a tangent vector on the traveling twisty configuration (which is quite an extraordinary twisty thing).

The light ray associated with (Z^0, Z^1, Z^2, Z^3) is the vector $(1, x, y, z)$ such that
$\begin{pmatrix} 1+z & x+iy \\ x-iy & 1-z \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
The determinant must be 0, so $x^2 + y^2 + z^2 = 1$.

Solving for x, y and z, the light ray is

$$\left(1, \frac{|Z^3|^2 - |Z^2|^2}{|Z^2|^2 + |Z^3|^2}, \frac{-Z^2\overline{Z^3} - \overline{Z^2}Z^3}{|Z^2|^2 + |Z^3|^2}, \frac{-Z^2\overline{Z^3} + \overline{Z^2}Z^3}{i(|Z^2|^2 + |Z^3|^2)}\right).$$

Any twistor, with 0 norm or not, has a light ray in a real direction!

To find the direction that the twistor is pointing, look at the direction of twistors (Y^0, Y^1, Y^2, Y^3) orthogonal to (Z^0, Z^1, Z^2, Z^3) as $(Y^0, Y^1, Y^2, Y^3) \to \infty$, i.e. as $Y^2, Y^3 \to 0$. At infinity, the twistor $(-\overline{Z^3}, \overline{Z^2}, 0, 0)$ is orthogonal to (Z^0, Z^1, Z^2, Z^3). Suppose $(-\overline{Z^3} + \epsilon^0, \overline{Z^2} + \epsilon^1, \epsilon^2, \epsilon^3)$, where the ϵ's are small, is also a null twistor and orthogonal to (Z^0, Z^1, Z^2, Z^3). Then $(\epsilon^0, \epsilon^1, \epsilon^2, \epsilon^3)$ is orthogonal to (Z^0, Z^1, Z^2, Z^3) and to $(-\overline{Z^3}, \overline{Z^2}, 0, 0)$. This is satisfied by taking (ϵ^2, ϵ^3) proportional to (Z^2, Z^3), so $(\epsilon^0, \epsilon^1, \epsilon^2, \epsilon^3)$ is orthogonal to $(-\overline{Z^3}, \overline{Z^2}, 0, 0)$, and then ϵ^0 and ϵ^1 can be chosen so that $(\epsilon^0, \epsilon^1, \epsilon^2, \epsilon^3)$ is orthogonal to (Z^0, Z^1, Z^2, Z^3).

So the light ray at infinity is in the same direction as the light ray belonging to (Z^0, Z^1, Z^2, Z^3). That means that the light ray at the center of the twisty configuration is pointing in the opposite direction from (Z^0, Z^1, Z^2, Z^3)'s light ray, and the twisty thing is traveling in the same direction as the light ray of (Z^0, Z^1, Z^2, Z^3).

So now we know enough about how it all works, to find the center of the twistor configuration. But first, I'll find the center of the circle for a twistor with simple coordinates. Which circle, you ask? Since the traveling twisty thing is entirely composed of circles? The circle, I mean, which is in a plane $\perp$ to the direction the twistor is traveling.

The twistor $(0, Z^1, 0, 1)$ has its light ray pointing in the z direction, and the center of its circle is on the z-axis, since $(0, 0, 1, 0)$ is orthogonal to it, and $(0, 0, 1, 0)$ is a twistor through the origin pointing in the $-z$ direction.

The circle for $(0, Z^1, 0, 1)$ is made of twistors with light rays orthogonal to the z-axis. A twistor $(\overline{Z^1}+ki, -\overline{Z^1}, 1, 1)$, with k real, has norm 0 and its light ray points in the $-x$ direction. Also it's orthogonal to $(0, Z^1, 0, 1)$, so if k is chosen right, it'll be part of the circle.

In Exercise 33.11) I gave formulas for the x, y, z coordinates at $t = 0$ for a twistor with norm 0. From those formulas, the location for $(\overline{Z^1}+ki, -\overline{Z^1}, 1, 1)$ is

$$(t, x, y, z) = \left(0, \frac{k}{\sqrt{2}}, \frac{-(\overline{Z^1}+Z^1)}{\sqrt{2}}, \frac{k}{\sqrt{2}} + \frac{i}{\sqrt{2}}(Z^1 - \overline{Z^1})\right).$$

The circle is where the distance for a twistor with tangent vector in the $-x$ direction is closest to the z-axis (see the expression for the stereographic projection of the tangent vector of a Clifford parallel, in Exercise 33.2) That's when $k = 0$, so $(\overline{Z^1}, -\overline{Z^1}, 1, 1)$ is on the circle, and it has coordinates $(t, x, y, z) = \left(0, 0, \frac{-(\overline{Z^1}+Z^1)}{\sqrt{2}}, \frac{i}{\sqrt{2}}(Z^1 - \overline{Z^1})\right)$.

Since the circle is centered on the z-axis, its center is at $(t, x, y, z) = \left(0, 0, 0, \frac{i}{\sqrt{2}}(Z^1 - \overline{Z^1})\right)$.

Once you know the coordinates of the center of the circle for a twistor pointing in the z direction, with its central ray going through the origin, you can find the coordinates of the center for an arbitrary twistor. Translations and spacelike rotations just transform the whole twisty configuration of circles in the same way. If the origin is displaced by $\vec{q}$, a twistor (ω, π) turns into $(\omega - i\mathbf{q}, \pi)$, where $\mathbf{q}$ is the position matrix for $\vec{q}$, as defined in Sec. 33.6. If (ω_y, π_y) is orthogonal to (ω, π), then $(\omega_y - i\mathbf{q}, \pi_y)$ is orthogonal to $(\omega - i\mathbf{q}, \pi)$. So the whole twisty configuration is translated by $-\vec{q}$ also.

Similarly, a twistor can be rotated in space by multiplying it by a matrix T, so (ω, π) becomes $(T\omega, T\pi)$. If the twistor is incident at $\vec{r}$, then $\omega = i\mathbf{r}\pi$, where $\mathbf{r}$ is the position matrix for $\vec{r}$. Since $T\omega = iT\mathbf{r}T^{-1}T\pi$, $\mathbf{r}$ is transformed to $T\mathbf{r}T^{-1}$. $T\mathbf{r}T^{-1}$ is also hermitian, so it can also be interpreted as a position matrix. The determinant of a position matrix $\begin{pmatrix} z & x+iy \\ x-iy & -z \end{pmatrix}$ is just minus the square of the Euclidean distance $x^2+y^2+z^2$, and $\det(T\mathbf{r}T^{-1}) = \det(\mathbf{r})$. So this is a distance- preserving transformation of Minkowski space which fixes the origin, i.e. a rotation. Actually, maybe it could also invert Minkowski space.

What does he mean by the location of the center of the circle not Lorentz transforming right? If you take the history of this location by itself, as a path in spacetime, it *has* to transform right, because it's just a null geodesic in spacetime, and it Lorentz transforms into another null geodesic. Not Lorentz transforming right has to involve the connection of this geodesic with the twistor.

What I think he means is that a twistor is associated with a plane in $\mathbb{CM}$. The Lorentz transform can be extended to $\mathbb{CM}$, so it transforms that plane. The plane has a unique associated twistor in $\mathbb{PT}$, so the Lorentz transform is defined on $\mathbb{PT}$. There's a null geodesic defined by the transformed twistor. And this null geodesic, he's saying, is not the same as the Lorentz transform of the geodesic defined by the original twistor.

So, give the twistor $(0, Z^1, 0, 1)$ a Lorentz boost in the x direction:
$t' = (t - vx)\beta,\ x' = (x - vt)\beta$, where $\beta = \frac{1}{\sqrt{1-v^2}}$.
Then the transformed geodesic is in the $(t, x, y, z) = (1, -v, 0, \sqrt{1-v^2})$ direction. The center of the twistor's circle is still at $\left(0, 0, 0, \frac{i}{\sqrt{2}}(Z^1 - \overline{Z^1})\right)$ in the primed coordinate system.

The complex plane associated with $(0, Z^1, 0, 1)$ is the set of points (t, x, y, z) such that

$$\begin{pmatrix} 0 \\ Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In the primed coordinate system, this is the set of points (t', x', y', z') with
$\beta v t' + \beta x' + iy' = 0$ and $\frac{i}{\sqrt{2}}\left(\beta(t' + vx') - z'\right) = Z^1$.
Rearranging these equations, you get

$$\frac{i}{\sqrt{2}}\left(\sqrt{1-v^2}\, x' + vz' + iy'\right) = -vZ^1, \text{ and}$$

$$\frac{i}{\sqrt{2}}\left(t' + vx' - \sqrt{1-v^2}\, z'\right) = \sqrt{1-v^2} Z^1.$$

If $\mathbb{CM}$ is rotated around the origin in the xz plane, by
$x'' = \sqrt{1-v^2}\, x' + vz'$,
$z'' = \sqrt{1-v^2}\, z' - vx'$

this is the plane for the twistor

$$\begin{pmatrix} -vZ^1 \\ \sqrt{1-v^2}\, Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} t''+z'' & x''+iy'' \\ x''-iy'' & t''-z'' \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which points in the z direction.

This twistor can be translated so its central line passes through the origin, by
$x''' = x'' - \mathrm{Re}(i\sqrt{2}\, vZ^1)$
$y''' = y'' - \mathrm{Im}(i\sqrt{2}\, vZ^1)$

$$\text{since } \begin{pmatrix} -vZ^1 \\ 0 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & i\sqrt{2}\, vZ^1 \\ -i\sqrt{2}\, v\overline{Z^1} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

So the center of its circle is (ta da!) at

$$(t'', x'', y'', z'') = \left(0,\ \mathrm{Re}(i\sqrt{2}\, vZ^1),\ \mathrm{Im}(i\sqrt{2}\, vZ^1),\ \frac{i}{\sqrt{2}}\sqrt{1-v^2}\,(Z^1 - \overline{Z^1})\right).$$

Translating back to the primed coordinates is a rotation in x and z which leaves the y coordinate alone, so it's clear that if $\mathrm{Re}\,(Z^1) \neq 0$, the center of the circle for the Lorentz transformed twistor isn't the same as the Lorentz transform of the center of the circle for the original twistor.

Roger Penrose must have been sad when he realized this!

But, if $\mathrm{Re}\,(Z^1) = 0$, i.e. the helicity of the twistor is 0, the center does transform back to $(0, 0, 0, i\sqrt{2}\, Z^1)$. For non-spinning particles, apparently the center does represent the particle's motion.

For a Lorentz boost of $(0, Z^1, 0, 1)$ in the z direction, the center of the transformed twistor is the same as the Lorentz transform of the center of the original twistor. If
$t' = (t - vz)\beta$
$z' = (z - vt)\beta$

then the transformed center is at

$$\left(0, 0, 0, \frac{i}{\sqrt{2}}\sqrt{\frac{1+v}{1-v}}(Z^1 - \overline{Z^1})\right).$$

Laura!