Exercise 33.2

Compactified Minkowski space has the spacetime track of a light-ray identified into a circle, so that the $t = \infty$ end is identified with the $t = -\infty$ end. A planar wavefront, with the plane $\perp$ to the direction the wave is traveling in, is identified to a single point at $\mathcal{I}^+$ and $\mathcal{I}^-$.

Why is the topology of compactified Minkowski space $S^3 \times S^1$?

You can see this by looking at a constant-time slice. It's topologically $S^3$ because it's Euclidean 3-space identified at infinity to the point $i^o$.

For each point in the constant-time slice, we need to assign a direction in 3-space. The spacetime path of a light ray going through that point in that direction is the $S^1$ fiber over that point in $S^3$.

To show the space is $S^3 \times S^1$ the light rays must:

Never cross

Go in all directions in 3-space

Only one light ray from a given planar wavefront can be chosen.

And there should be continuous projection maps from compactified Minkowski space to the $S^3$ and $S^1$ components of the product.

Finding those directions isn't easy! I sort of looked it up. It's called a Robinson congruence.

Remember the Clifford parallels in $S^3$, introduced in sec. 15.4? How the Robinson congruence works is that the Clifford parallels are stereographically projected into Euclidean 3-space. The tangent vectors to the Clifford parallels give you the directions of light rays in a $t = 0$ slice of Minkowski space!

In $S^3 = \{(w, x, y, z) : w^2 + x^2 + y^2 + z^2 = 1\}$, a Clifford parallel is

$$(w, x, y, z) : \frac{w + ix}{y + iz} \text{ is constant.}$$

If a given $(w, x, y, z)$ is on a parallel, you can find the other points on the parallel by multiplying $w + ix$ and $y + iz$ by $e^{i\theta}$. The tangent vector to the parallel is $(-x, w, -z, y)$.

$(w, x, y, z)$ is stereographically projected into Euclidean 3-space $(0, x, y, z)$ by drawing a line from $(-1, 0, 0, 0)$ to $(w, x, y, z)$ and finding the $w = 0$ point on the line.

The stereographic projection of the tangent vector of a Clifford parallel at $(w, x, y, z)$ is

$$\left( \frac{1 + x^2 - (y^2 + z^2)}{1 + x^2 + y^2 + z^2}, \frac{2(xy - z)}{1 + x^2 + y^2 + z^2}, \frac{2(xz + y)}{1 + x^2 + y^2 + z^2} \right)$$

First, I'll show that these tangent vectors are onto all directions in 3-space. Actually, they aren't quite onto; the direction $(-1, 0, 0)$ is saved for spacelike infinity $i^o$.

For a given $x$, the $y : z$ ratio can be chosen so that the $\frac{\partial}{\partial y} : \frac{\partial}{\partial z}$ ratio of the 2nd and 3rd components of the tangent vector is anything from $-\infty$ to $\infty$.

If $x = 0$ this is obvious. Suppose $x \neq 0$. If $y = 0$ the ratio is $-1/x$. Otherwise let $z = \alpha y$. Then

$$\frac{\partial/\partial y}{\partial/\partial z} = \frac{x-\alpha}{\alpha x + 1}.$$

As $\alpha$ varies from $-\infty$ to $\infty$ (not including infinite values), $(x-\alpha)/(\alpha x+1)$ takes on all values including $-\infty$ and $\infty$, except for $-1/x$.

If $y^2 + z^2 = \gamma x^2$, the $\partial/\partial x$ component of the tangent vector is

$$\frac{1 + (1-\gamma)x^2}{1 + (1+\gamma)x^2}.$$

$\gamma$ varies independently of the $y : z$ ratio. As $\gamma$ varies from 0 to $\infty$ (not including $\infty$), $\partial/\partial x$ varies from 1 to approaching -1. If $x = 0$, $\partial/\partial x = 1$.

A value of $\partial/\partial x$ and a ratio of $\partial/\partial y$ to $\partial/\partial z$ determine the tangent vector, except for the sign of $\partial/\partial y$ and $\partial/\partial z$. The sign can be changed by changing the sign of $y$ and $z$.

So it's onto except for $(-1, 0, 0)$, which is used for $i^o$.

Why do the light rays never cross?

If you use derivatives to find the direction in which the tangent vector

$$\left(\frac{1 + x^2 - (y^2+z^2)}{1 + x^2 + y^2 + z^2}, \frac{2(xy - z)}{1 + x^2 + y^2 + z^2}, \frac{2(xz + y)}{1 + x^2 + y^2 + z^2}\right)$$

doesn't change, you find it's $(1 + x^2, xy - z, xz + y)$. That means that if the light rays travel a time $t$ in the tangent vector direction, it's as if the assignment of tangent vectors to points has been displaced a distance of $t$ in the $x$ direction! Since

$$\left(\frac{1 + x^2 - (y^2+z^2)}{1 + x^2 + y^2 + z^2}, \frac{2(xy - z)}{1 + x^2 + y^2 + z^2}, \frac{2(xz + y)}{1 + x^2 + y^2 + z^2}\right) -$$

$$\frac{2}{1 + x^2 + y^2 + z^2}(1 + x^2, xy - z, xz + y) = (-1, 0, 0).$$

The whole pattern of light rays moves in the positive $x$ direction. Since the whole pattern is just being moved, the light rays never cross!

For a given distance from the origin, there are at most two points with a given tangent vector, and they are joined by a vector in the direction $(1 + x^2, xy - z, xz + y)$. So the locus of points with a given tangent vector is just one line in the direction $(1 + x^2, xy - z, xz + y)$.

Also, $(1 + x^2, xy - z, xz + y)$ is never $\perp$ to the tangent vector in the $(1 + x^2 - (y^2 + z^2), xy - z, xz + y)$ direction. So the light rays belonging to distinct points in a $t = 0$ slice of Minkowski space don't meet at $\infty$.

For the projection onto $S^1$ you might think - just take the time! That doesn't work though, because for the circle at $i^o$ the time is infinite for the whole circle. So, by inspiration from the case with only one space dimension, what you can do is to define the projection to be 0 for a plane containing the tangent vector $(1,-1,0,0)$ belonging to $i^o$, $(0,0,1,0)$ and $(0,0,0,1)$ and going through the origin. For other points in the compactified Minkowski space, the projection is defined to be the signed distance from this plane, with $\pm\infty$ identified. This projection does work for the circle associated with $i^o$.

For the projection onto $S^3$, just find the light ray that goes through a point in Minkowski space and trace it to its point of intersection with the plane containing the vectors $(1,-1,0,0)$, $(0,0,1,0)$ and $(0,0,0,1)$ and going through the origin. This map can be made continuous at $i^o$ by defining the topology on the compactified Minkowski space so that the open sets are the inverse images of open sets in $S^3$.

Now if somebody could explain to me why Clifford parallels projected into 3-space have these magical properties, I'd appreciate it!

What would change if there were an odd number of spacetime dimensions?

If there's an even number of spacetime dimensions, you can define a field of tangent vectors in a constant-time slice in the same way. You would have complex numbers $z_1, ..., z_n$ with $|z_1|^2 + ... + |z_n|^2 = 1$, and the circles on the sphere $S^{2n-1}$ would be $e^{i\theta}(z_1, ..., z_n)$, and the tangents to the circles would be stereographically projected into $2n-1$ dimensional Euclidean space. Maybe this tangent vector field even has the same nice properties as it does for $S^1$ and $S^3$.

With an odd no. of spacetime dimensions, you couldn't do this.

Also, you can't have a Killing vector field defined on a spacelike slice of the spacetime, because there isn't a continuous non-zero vector field on $S^{2n}$.

*Laura*