Exercise 33.24 The massless free-field equations for a photon (spin 1),

$$\nabla^{AA'}\psi_{AB} = 0 \tag{1}$$

and

$$\nabla^{AA'}\psi_{A'B'} = 0 \tag{2}$$

turn into Maxwell's equations in empty space if
$\psi_{00} = E_x - B_y - iE_y - iB_x$,
$\psi_{01} = -E_z + iB_z$
$\psi_{11} = -E_x - B_y - iE_y + iB_x$.

I left out a factor of 2, it doesn't matter because the righthand sides of (1) and (2) are 0.
There's a typo in the exercise, it should be $\psi_{01} = -C_3$, not $-iC_3$.

$\nabla^{AA'}$ is derived from $\nabla^a = -\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ the same way that $r^{AA'}$ is derived from (t, x, y, z). So

$$\nabla^{AA'} = \begin{pmatrix} -\frac{\partial}{\partial t} + \frac{\partial}{\partial z} & \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} & -\frac{\partial}{\partial t} - \frac{\partial}{\partial z} \end{pmatrix}.$$

I just worked out $\nabla^{AA'}\psi_{AB} = 0$.

If $A' = 0$ and $B = 0$, you get

$$-\frac{\partial \psi_{00}}{\partial t} + \frac{\partial \psi_{00}}{\partial z} + \frac{\partial \psi_{10}}{\partial x} - i\frac{\partial \psi_{10}}{\partial y} = 0, \text{ or}$$

$$-\frac{\partial E_x}{\partial t} + \frac{\partial B_y}{\partial t} + \frac{\partial E_x}{\partial z} - \frac{\partial B_y}{\partial z} - \frac{\partial E_z}{\partial x} + \frac{\partial B_z}{\partial y} = 0 \text{ and} \tag{3}$$

$$\frac{\partial E_y}{\partial t} + \frac{\partial B_x}{\partial t} - \frac{\partial E_y}{\partial z} - \frac{\partial B_x}{\partial z} + \frac{\partial B_z}{\partial x} + \frac{\partial E_z}{\partial y} = 0, \tag{4}$$

separating real and imaginary parts.

If $A' = 0$ and $B = 1$, you get

$$-\frac{\partial \psi_{01}}{\partial t} + \frac{\partial \psi_{01}}{\partial z} + \frac{\partial \psi_{11}}{\partial x} - i\frac{\partial \psi_{11}}{\partial y} = 0, \text{ or}$$

$$\frac{\partial E_z}{\partial t} - \frac{\partial E_z}{\partial z} - \frac{\partial E_x}{\partial x} - \frac{\partial B_y}{\partial x} - \frac{\partial E_y}{\partial y} + \frac{\partial B_x}{\partial y} = 0 \text{ and} \tag{5}$$

$$-\frac{\partial B_z}{\partial t}+\frac{\partial B_z}{\partial z}-\frac{\partial E_y}{\partial x}+\frac{\partial B_x}{\partial x}+\frac{\partial E_x}{\partial y}+\frac{\partial B_y}{\partial y}=0. \tag{6}$$

If $A'=1$ and $B=0$, you get

$$\frac{\partial \psi_{00}}{\partial x}+i\frac{\partial \psi_{00}}{\partial y}-\frac{\partial \psi_{10}}{\partial t}-\frac{\partial \psi_{10}}{\partial z}=0, \text{ or}$$

$$\frac{\partial E_x}{\partial x}-\frac{\partial B_y}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial B_x}{\partial y}+\frac{\partial E_z}{\partial t}+\frac{\partial E_z}{\partial z}=0 \text{ and} \tag{7}$$

$$-\frac{\partial E_y}{\partial x}-\frac{\partial B_x}{\partial x}+\frac{\partial E_x}{\partial y}-\frac{\partial B_y}{\partial y}-\frac{\partial B_z}{\partial t}-\frac{\partial B_z}{\partial z}=0. \tag{8}$$

If $A'=1$ and $B=1$, you get

$$\frac{\partial \psi_{01}}{\partial x}+i\frac{\partial \psi_{01}}{\partial y}-\frac{\partial \psi_{11}}{\partial t}-\frac{\partial \psi_{11}}{\partial z}=0, \text{ or}$$

$$-\frac{\partial E_z}{\partial x}-\frac{\partial B_z}{\partial y}+\frac{\partial E_x}{\partial t}+\frac{\partial B_y}{\partial t}+\frac{\partial E_x}{\partial z}+\frac{\partial B_y}{\partial z}=0 \text{ and} \tag{9}$$

$$\frac{\partial B_z}{\partial x}-\frac{\partial E_z}{\partial y}+\frac{\partial E_y}{\partial t}-\frac{\partial B_x}{\partial t}+\frac{\partial E_y}{\partial z}-\frac{\partial B_x}{\partial z}=0. \tag{10}$$

So equations $(7)-(5)$ give you $\nabla \cdot E=0$,
eqs. $(6)-(8)$ give $\nabla \cdot B=0$,
eqs. $(3)-(9)$, $(4)+(10)$, $(5)+(7)$ give $\frac{\partial E}{\partial t}=\nabla \times B$,
eqs. $(4)-(10)$, $(3)+(9)$, $(6)+(8)$ give $-\frac{\partial B}{\partial t}=\nabla \times E$.

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