Exercise 33.5

Let (a,b,c,d,e,f) be a vector from the origin, lying in the cone $w^2+t^2-x^2-y^2-z^2-v^2=0$, so that $a^2+b^2-c^2-d^2-e^2-f^2=0$. Minkowski space is the intersection of the cone with the plane $\alpha w+\beta t+\gamma x+\delta y+\epsilon z+\theta v=k$. The line generated by (a,b,c,d,e,f) intersects this plane in the point

$$F(a,b,c,d,e,f)=\frac{k}{\alpha a+\beta b+\gamma c+\delta d+\epsilon e+\theta f}(a,b,c,d,e,f).$$

The conformal metric in the Minkowski space is independent of the particular plane of intersection, as long as it doesn't pass through the origin. That means that if you take two vectors $\vec{n}=(n_1,n_2,n_3,n_4,n_5,n_6)$ and $\vec{m}=(m_1,m_2,m_3,m_4,m_5,m_6)$ from (a,b,c,d,e,f), the dot product $\partial F/\partial\vec{n}\cdot\partial F/\partial\vec{m}$ is C$\vec{n}\cdot\vec{m}$, and C is a function only of the plane of intersection.

If you work it out,
$\partial F/\partial\vec{n}\cdot\partial F/\partial\vec{m}=$

$$\frac{k^2}{(\alpha a+\beta b+\gamma c+\delta d+\epsilon e+\theta f)^4}[(n_1m_1+n_2m_2-n_3m_3-n_4m_4-n_5m_5-n_6m_6)(\alpha a+\beta b+\gamma c+\delta d+\epsilon e+\theta f)^2$$

$$-(\alpha m_1+\beta m_2+\gamma m_3+\delta m_4+\epsilon m_5+\theta m_6)(an_1+bn_2-cn_3-dn_4-en_5-fn_6)(\alpha a+\beta b+\gamma c+\delta d+\epsilon e+\theta f)$$

$$-(\alpha n_1+\beta n_2+\gamma n_3+\delta n_4+\epsilon n_5+\theta n_6)(am_1+bm_2-cm_3-dm_4-em_5-fm_6)(\alpha a+\beta b+\gamma c+\delta d+\epsilon e+\theta f)$$

$$+(a^2+b^2-c^2-d^2-e^2)(\alpha n_1+\beta n_2+\gamma n_3+\delta n_4+\epsilon n_5+\theta n_6)(\alpha m_1+\beta m_2+\gamma m_3+\delta m_4+\epsilon m_5+\theta m_6)].$$

Since $\vec{n}$ and $\vec{m}$ are tangent to the cone $w^2+t^2-x^2-y^2-z^2-v^2=0$,
$an_1+bn_2-cn_3-dn_4-en_5-fn_6$ and $am_1+bm_2-cm_3-dm_4-em_5-fm_6$ are 0. Also $a^2+b^2-c^2-d^2-e^2-f^2=0$. So the dot product simplifies to

$$\partial F/\partial\vec{n}\cdot\partial F/\partial\vec{m}=\frac{k^2}{(\alpha a+\beta b+\gamma c+\delta d+\epsilon e+\theta f)^2}\vec{n}\cdot\vec{m},$$

which means that the conformal metric is independent of the plane you choose to intersect with.

How does the limit of Minkowski space at ∞ work? Suppose the plane of intersection is $w-v=1$. Then a finite point in Minkowski space is

$$\left(\frac{1+c^2+d^2+e^2-b^2}{2},b,c,d,e,\frac{-1+c^2+d^2+e^2-b^2}{2}\right),$$

applying the two rules $w-v=1$ and $w^2+t^2-x^2-y^2-z^2-v^2=0$ to the coordinates.

A light ray going through (b,c,d,e) is (b',c',d',e'), where
$b'=b+t$
$c'=c+\alpha_1 t$

$d' = d + \alpha_2 t$
$e' = e + \alpha_3 t,$
$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1.$

If you take the limit as $t \to \infty$, the light ray approaches the line

$$(2(\alpha_1 c + \alpha_2 d + \alpha_3 c - b, 1, \alpha_1, \alpha_2, \alpha_3, 2(\alpha_1 c + \alpha_2 d + \alpha_3 c - b)),$$

which is in the plane $w - v = 0$.

So if two points (b, c_1, d_1, e_1) and (b, c_2, d_2, e_2) are in a constant-time slice of Minkowski space, light rays through them go to the same limit point iff they are in the same direction and $(c_1 - c_2, d_1 - d_2, e_1 - e_2)$ is $\perp$ to $(\alpha_1, \alpha_2, \alpha_3)$. So, all the light rays in a plane wavefront go to the same limit point.

The limit point at spacelike ∞ and timelike ∞ is $(1, 0, 0, 0, 1)$. Not all the lines in the $w - v = 0$ plane are used up, but that's OK. $\Diamond$

Laura