Exercise 33.8 The celestial sphere is a conformal manifold in spacetime with any number of space and time dimensions.

What is the celestial sphere? It's the 0-length vectors going through a point. In our spacetime, it is the set of light rays that might arrive at a given point in spacetime. If the spacetime has $n$ time dimensions and $m$ space dimensions, $x_1^2 + ... + x_n^2 - (x_{n+1}^2 + ... + x_{n+m}^2) = 0$ for a 0-length vector $(x_1, ..., x_n, x_{n+1}, ..., x_{n+m})$.

And what's a conformal manifold? It's a manifold that doesn't necessarily have a metric, but there's a well-defined angle between two tangent vectors at a point.

You can define a metric on the null vectors by looking at their intersection with a hyperplane that doesn't go through the central point, and using the spacetime metric in the hyperplane. So we want to show that with this definition, the angle between two vectors doesn't depend on the hyperplane. That means that you can define a conformal metric on the space of 0-length vectors going through the point.

In this space, a 0-length vector is a point, not a vector! So a vector in this space describes an infinitesimal change in the 0-length vector, so that the new vector still has 0 length.

For simplicity take the origin as the central point. Given a hyperplane $b_1x_1 + ... + b_{m+n}x_{m+n} = k$ with $k \neq 0$, the vector $(x_1, ..., x_{m+n})$ intersects the hyperplane in the point

$$f(x_1, ..., x_{m+n}) = \frac{k}{\sum b_i x_i}(x_1, ..., x_{m+n}).$$

$\vec{v} = (v_1, ..., v_{m+n})$ is a tangent vector to the manifold if $v_1x_1 + ... + v_nx_n - (v_{n+1}x_{n+1} + ... + v_{n+m}x_{n+m}) = 0$.

$\vec{v}$ is projected to the vector

$$\frac{\partial f}{\partial \vec{v}} = \frac{k}{(\sum b_i x_i)^2}\Big(v_1\big(\sum b_i x_i\big) - \big(\sum b_i v_i\big)x_1, ..., v_{m+n}\big(\sum b_i x_i\big) - \big(\sum b_i v_i\big)x_{m+n}\Big).$$

So,

$$\frac{\partial f}{\partial \vec{v}} \cdot \frac{\partial f}{\partial \vec{w}} = \frac{k^2}{(\sum b_i x_i)^4}\Big( [v_1w_1 + ... + v_nw_n - (v_{n+1}w_{n+1} + ... + v_{n+m}w_{n+m})] \big(\sum b_i x_i\big)^2$$
$$-\big(\sum b_i v_i\big)\big(x_1w_1 + ... + x_nw_n - (x_{n+1}w_{n+1} + ... + x_{n+m}w_{n+m})\big)$$
$$-\big(\sum b_i w_i\big)\big(x_1v_1 + ... + x_nv_n - (x_{n+1}v_{n+1} + ... + x_{n+m}v_{n+m})\big)$$
$$+\big(\sum b_i v_i\big)\big(\sum b_i w_i\big)\big(x_1^2 + ... + x_n^2 - (x_{n+1}^2 + ... + x_{n+m}^2)\big)\Big) =$$
$$\frac{k^2}{(\sum b_i x_i)^2}\big(v_1w_1 + ... + v_nw_n - (v_{n+1}w_{n+1} + ... + v_{n+m}w_{n+m}\big).$$

The angle between $\partial f/\partial \vec{v}$ and $\partial f/\partial \vec{w}$ is

$$\frac{\frac{\partial f}{\partial \vec{v}} \cdot \frac{\partial f}{\partial \vec{w}}}{\sqrt{\frac{\partial f}{\partial \vec{v}} \cdot \frac{\partial f}{\partial \vec{v}} \frac{\partial f}{\partial \vec{w}} \cdot \frac{\partial f}{\partial \vec{w}}}}$$

which is the same as the angle

$$\frac{\vec{v} \cdot \vec{w}}{\sqrt{|\vec{v}|^2 |\vec{w}|^2}},$$

so it's independent of the hyperplane chosen.

The angle could be infinite, though. For example, suppose there are 2 space dimensions and 2 time dimensions, and $(x_1, ..., x_4)$ is a 0-length vector, so $x_1^2 + x_2^2 = x_3^2 + x_4^2$, and $\vec{v} = (x_2, -x_1, x_4, -x_3)$ is tangent to $(x_1, ...x_4)$. Then the angle between tangent vector $\vec{w} = (x_2, -x_1, 0, 0)$ and $\vec{v}$ is infinite.

The angle could also be complex if the norm of $\vec{v}$ is negative.

*Laura*