2006 IMO problem 6:

Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P. Show that the sum of the areas assigned to the sides of P is at least twice the area of P.

First assume that no sides are parallel. It's easy to adjust the following if some sides are parallel.

The sides of the polygon can be ordered by going counterclockwise around the polygon. Call the vertices ordered this way $\{v_i\}$.

The sides can also be ordered by the angle of the side, mod π: so the side $S_i = v_{s(i)}v_{s(i)+1}$ is at angle θ_i mod π. See fig. 1.

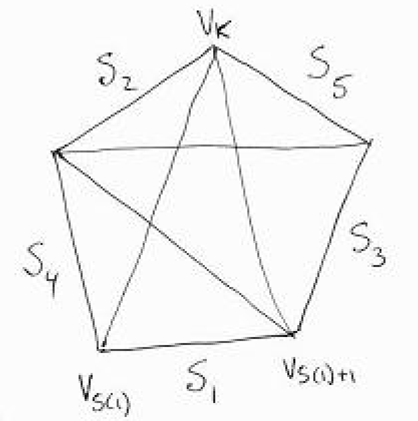

Fig. 1

Now let's consider the maximum area triangles based on the sides, in the order of increasing angle mod π. The triangle based on S_1 has its 3rd vertex at the first vertex $v_{s(k)}$ on the opposite side of the polygon such that $\theta_1 < \theta_k$ mod π.

If S_2 is adjacent to S_1, then the triangle based on S_2 also has its 3rd vertex at $v_{s(k)}$. So, the two triangles share a side $v_{s(1)+1}v_k$.

If S_2 is on the other side of the polygon from S_1, then the triangle based on S_2 has its 3rd vertex at $v_{s(1)+1}$. So in this case also, the two triangles share a side $v_{s(1)+1}v_k$.

These maximum area triangles can be translated so their matching sides are together, in the order of increasing angle mod π, with the 3rd vertices all translated to the origin. The maximum area triangle based at the last side S_n contains the line $v_{s(1)}v_k$, so the angles at the 3rd vertices sum up to π. Going around the polygon a second time, we assemble a new polygon from the maximum area triangles, which contains each side of the original polygon twice, at opposite sides, so it has 2-fold rotational symmetry. See fig. 2.

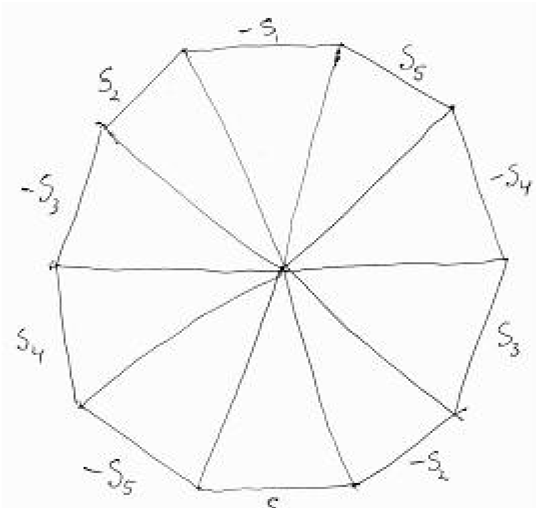

Fig.2

What to do with this nifty polygon? We're asked to prove that it has area ≥ 4 times the area of the original polygon. Scaling the lengths down by a factor of 2, we're asked to show that it has area $\geq$ the area of the original polygon. This seems "obvious", because the new polygon has the same perimeter as the first one, but twice as many sides, so it looks more like a circle. But how to prove it?

Each side of the original polygon is copied twice in the new polygon, scaled down by 2. Going counterclockwise around the new polygon, label each side as the first instance or the second instance of the side, which describes the new polygon as a kind of superposition of two copies of the original polygon, with one copy rotated by π.

Call the vectors of the original polygon $w_1, ..., w_n$. The vectors of the new polygon are $\frac{w_1}{2}, ..., \frac{w_n}{2}, -\frac{w_1}{2}, ..., -\frac{w_n}{2}$.

But we can make a polygon using any real t, from sides $tw_1, ..., tw_n, (t-1)w_1, ..., (t-1)w_n$, with these sides ordered in the same way that they were in the new polygon. If $t = \frac{1}{2}$, we get the new polygon. If $t = 0$ or 1, we get the original polygon.

The area of this polygon is a quadratic polynomial in t. If $1 - t$ is substituted for t, the area stays the same. So the quadratic polynomial is symmetric about $1/2$. If the area $\to -\infty$ as $t \to \infty$ then the area is maximum at $t = 1/2$, and it's proved.

So, we need to show that the area of the polygon with sides $S_1, ..., S_n$, which are the edges of the original polygon ordered by the angle mod π, given the vector directions from the original polygon, is negative.

Kai Neergård came up with a nice proof of this:

First orient this polygon so that the slope changes from positive to negative across a particular vertex, and place this vertex at the origin. Going around the polygon, the slopes of the edges increase from one edge to the next, except across the vertex at the origin.

Let the equation of the line for S_i be $y = a_i x + d_i$. Let (x_i, y_i) be the coordinates of the common vertex of S_i and S_{i+1}, where the indices are mod n. Then the oriented area of this polygon is given by

$$\sum_i (d_i + a_i \frac{x_{i-1} + x_i}{2})(x_{i-1} - x_i) =$$

$$\sum_i d_i(x_{i-1} - x_i) + a_i \frac{x_{i-1}^2 - x_i^2}{2} =$$

$$\sum_i (d_{i+1} - d_i)x_i + \frac{(a_{i+1} - a_i)x_i^2}{2} =$$

$$\sum_i -(a_{i+1} - a_i)x_i^2 + \frac{(a_{i+1} - a_i)x_i^2}{2} =$$

$$-\sum_i \frac{(a_{i+1} - a_i)x_i^2}{2}.$$

Since the slope increases at all the vertices except the one at the origin, the area is negative.

Alternatively, you can show that the area of the polygon in Fig. 2 must be $\geq$ the 4 times the area of the original polygon by observing that it's the Minkowski sum of the original polygon and the original polygon rotated by π. The Minkowski sum of two sets A and B is $\{a + b\}$, $a \in A, b \in B$.

Place the original polygon so that it extends in x from 0 to L. Let $f(x)$ be its thickness in y. Place the rotated version of the original polygon so it also extends in x from 0 to L. Then the thickness in y of the Minkowski sum at $2x$ is at least $f(x) + f(L - x)$. So the area of the Minkowski sum is at least $2 \int_0^L f(x) + f(L - x) dx$, which is at least 4 times the area of the original polygon.