Suppose you have n coins, and they are all the same weight except for one of them, which is either lighter or heavier than the rest.

You have a balance scale, the kind with two dishes hanging from either end of a rod that is supported in the middle. You need to figure out which is the odd coin, and whether it's lighter or heavier than the other coins. You only have the balance scale and the n coins to use in your weighings.

What is the minimum number of weighings in which you can be sure to figure out which is the odd coin, and whether it's lighter or heavier than the rest?

Answer :

You can code the weighings of each coin by -1, 0, 1, where -1 means the coin is on the LH side in a weighing, 0 means it's not used in a weighing, and 1 means it's on the RH side.

So, if you have k weighings, to each coin there's associated a vector of dimension k, and each coordinate is one of -1, 0, 1.

For example, with 3 coins and 2 weighings, the weighings could be described by (-1 1), (1 0), (0 -1).

That means that first you weigh 1 vs 2, then 3 vs 1.

The vectors for different coins have to be different, in order to discriminate between the coins. Also, the vector for one coin can't be the negative of the vector for another coin, because you have to discriminate between one coin being light and the other one being heavy. And because an equal number of coins have to be on each side in a weighing, the coin vectors have to add up to 0. So those are the constraints on the coin vectors.

This weighing scheme can be described by a matrix $W_2 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$.

Here, $W_2(i,j) = 1$ if (i,j) is a coin vector; $W_2(i,j) = -1$ if (i,j) is a the negative of a coin vector, i.e. the result of the weighings if that coin is lighter than the others. And $W_2(i,j) = 0$ if neither (i,j) nor $(-i,-j)$ is a coin vector.

Note that $W_2(i,j) \equiv i - j \bmod 3$.

The diagonal of W_2 has zeroes. Why?

(0,0) can't be used as a coin vector, because that would mean the coin isn't weighed at all, and you wouldn't know if it was heavy or light.

There can't be 3 coin vectors with ± 1 in the first coordinate, because the coin vectors have to sum up to 0. So (1 1) and (-1 -1) can be eliminated as coin vectors.

Now we can add another weighing.

For each of the 3 coin vectors of dimension 2, make 3 coin vectors of dimension 3, with the 3rd coordinate one of {-1, 0, 1}.

So we have 9 3-dimensional coin vectors which sum up to 0:

(-1 1 -1), (-1 1 0), (-1 1 1), (1 0 -1), (1 0 0), (1 0 1), (0 -1 -1), (0 -1 0), (0 -1 1).

The excluded vectors (0 0), (1 1) and (-1 -1) can be extended to create 3 more coin vectors of dimension 3: (-1 -1 0), (0 0 1), and (1 1 -1), which add up to 0.

So for 3 weighings, we have a $3 \times 3 \times 3$ array

$$W_3(i,j,-1) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix}; W_3(i,j,0) = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & -1 \end{pmatrix}; W_3(i,j,1) = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Note that $W_3(i,i,k) \equiv 2i + k \bmod 3$, and the diagonal of W_3 is again 0.

So in 3 weighings, you can find the answer for $(3^3 - 3)/2 = 12$ coins.

You can keep on adding weighings this way, and create a k-dimensional array W_k with indices $\{x_1, ..., x_k\} \in \{-1, 0, 1\}$ describing the weighing strategy for $(3^k - 3)/2$ coins.

If $x_1 \neq x_2$, then $W_k(x_1, x_2, ..., x_k) = x_1 - x_2 \bmod 3$, choosing a value from {-1, 0, 1}.

If $x_1 = ... = x_m$ and $x_{m+1} \neq x_m$ then $W_k(x_1, x_2, ..., x_k) = 2x_m + x_{m+1} \bmod 3$, choosing a value from {-1, 0, 1}.

if $x_1 = ... = x_k$ then $W_k(x_1, x_2, ..., x_k) = 0$.

So that gives a weighing strategy for n coins, if $n = (3^k - 3)/2$ for some k.

It's not too hard to convince yourself that if $(3^{k-1} - 3)/2 < n < (3^k - 3)/2$, then you can find the answer in k weighings.