The following comment in a book was puzzling to me : "Recall the Hodge dual of the Maxwell tensor F. We could imagine a 'dual' U(1) gauge connection that has $^*F$ as its bundle curvature rather than F."

But F_{ab} satisfies $dF_{ab} = 0$, which means that F_{ab} can be expressed as a derivative of a 1-form: $F_{ab} = \partial A_b/\partial x^a - \partial A_a/\partial x^b$.

and the vector potential A_a gives you a gauge connection $\nabla_a = \partial/\partial x^a - ieA_a$. So F_{ab} is the curvature of the connection, $\nabla_a\nabla_b - \nabla_b\nabla_a = -ie(\partial A_b/\partial x^a - \partial A_a/\partial x^b)$.

But $d^*F_{ab} = -\frac{^*J}{\epsilon_0}$. How can $^*F$ be a gauge curvature when $d^*F \neq 0$? If it were in empty space you could come up with a 1-form Z_a so that $^*F_{ab} = \partial Z_b/\partial x^a - \partial Z_a/\partial x^b$, that would give you an alternate bundle connection. Who knows what it would be a connection for!

Is the empty-space version of the Maxwell equations what you'd be using at the quantum level? Maybe all the interactions are being described explicitly so you wouldn't be using a charge-current vector. What do you think?