If you have comments about these comments, you can email me at pbbl@cyberspace.org. To get through my mail filter, include "glimful" in the Subject: line. I've tried to include all actual mathematical errors, but not all the typos.

-Laura

p. 41, Theorem D12: The proof of this theorem depends on Lemma D11, and the proof of that has an error, as Kenneth König notes. A proof that works can be found in Lars Hörmander's *An Introduction to Complex Analysis in Several Variables*, where it's Theorem 2.5.10.

p. 54, last line $O_D \to O_E$

p. 65, error in the proof of Corollary G6, in the step "the functions $\delta_{D,R}$ are a monotonically increasing sequence of continuous functions in D, and hence these functions converge uniformly on any compact subset of D."

Apparently the idea is to apply Dini's theorem. But Dini's theorem only applies if the limit function is continuous, and $d_{D,W}(z)$ isn't necessarily continuous.

For example, let D be $B(0;10) \subset \mathbb{C}$ with the set $\{z : \mathrm{Im}(z) = 0, \mathrm{Re}(z) \leq 0\}$ deleted. Then if $W = 1$, $d_{D,W}(z)$ isn't continuous.

However it is in fact true that $d_D(K) = d_D(\widehat{K}_D)$. If you rotate a polydisc with polyradius $I = (1, ..., 1)$ in all directions, it spins out a ball of radius $\sqrt{n}$. So

$$d_D(z) = \inf_{T \in U(n)} \delta_{T(D), \frac{I}{\sqrt{n}}}(T(z)),$$

where T is a unitary transformation on $\mathbb{C}^n$.

So

$$d_D(K) = \inf_{z \in K} \inf_{T \in U(n)} \delta_{T(D), \frac{I}{\sqrt{n}}}(T(z)) = \inf_{T \in U(n)} \inf_{z \in K} \delta_{T(D), \frac{I}{\sqrt{n}}}(T(z)) =$$

$$\inf_{T \in U(n)} \delta_{T(D), \frac{I}{\sqrt{n}}}(T(K)) = \inf_{T \in U(n)} \delta_{T(D), \frac{I}{\sqrt{n}}}(\widehat{T(K)}_{T(D)}) = \inf_{T \in U(n)} \delta_{T(D), \frac{I}{\sqrt{n}}}(T(\widehat{K}_D)) =$$

$$\inf_{T \in U(n)} \inf_{z \in \widehat{K}_D} \delta_{T(D), \frac{I}{\sqrt{n}}}(T(z)) = \inf_{z \in \widehat{K}_D} \inf_{T \in U(n)} \delta_{T(D), \frac{I}{\sqrt{n}}}(T(z)) = d_D(\widehat{K}_D).$$

p. 69 At the end of the first paragraph: "setting $\psi = \Psi|L$, it is evident that ψ is a C^∞ differential form..." But what if Ψ has terms involving $d\bar{z}_n$?

It turns out that the embedding map $L \to \mathbb{C}^n$ induces a map from differential forms on $\mathbb{C}^n$ to forms on L, which maps $d\bar{z}_n$ and z_n to 0. This map commutes with $\bar{\partial}$, which means that $\phi = \bar{\partial}\psi$.

p. 72 definition H1

Note that a Riemann domain M isn't compact. If M were compact, then $P(M)$ is compact. But P is an open map, since it's a local homeomorphism. So $P(M)$ is open.

p. 73 Definition H2: Why is $d_M(Z)$ the distance to the boundary of the Riemann domain? Why can't P fail to be injective if expanded to a larger neighborhood, so that there is no $B_M(Z;\epsilon)$ for $\epsilon > d_M(Z)$? I think it can't; maybe there's an easier proof, but here comes the

Long Lemming: If $Z \in M$, and U is an open neighborhood of Z such that $P|_U$ is a homeomorphism onto a ball $B(P(Z);r) \in \mathbb{C}^n$, and for every point $b \in$ boundary(B)
$\exists z \in$ boundary(U) such that $P(z) = b$, then $r < d_M(Z)$.

Proof: First, if $z_1 \in$ boundary(U), then $P(z_1) \in$ boundary(B). Suppose not. Then $P(z_1) \in B$. There is a sequence $\{u_1, ..., u_m, ...\} \subset U$ that tends to z_1. Since $\{P(u_1), ..., P(u_m), ...\}$ is contained in a compact subset of B, $\{u_1, ..., u_m, ...\}$ is contained in a compact subset of U, so $\{u_1, ..., u_m, ...\}$ has an accumulation point in U. Since M is Hausdorff, this is a contradiction.

So U contains all points in a connected component of $P^{-1}(B)$.

P is injective on boundary(U); to see this, suppose $z_1, z_2 \in$ boundary(U) with $P(z_1) = P(z_2)$. Let U_1, U_2 be neighborhoods containing z_1 and z_2 which are mapped homeomorphically by P to balls in $\mathbb{C}^n$ centered at $P(z_1) = P(z_2)$. Make U_1 and U_2 small enough so that $U_1 \cap U_2 = \phi$. $U_1 \cap P^{-1}(B)$ is connected because $P(U_1) \cap B$ is connected. There are elements of U in $U_1 \cap P^{-1}(B)$, since $z_1 \in$ boundary(U). Since U consists of all points in a connected component of $P^{-1}(B)$, $U_1 \cap P^{-1}(B) \subset U$. Also $U_2 \cap P^{-1}(B) \subset U$. Let $\{b_1, ..., b_m, ...\}$ be points in $B \cap P(U_1) \cap P(U_2)$ approaching $P(z_1)$. Then $P|_{U_1}^{-1}\{b_1, ..., b_m, ...\} \subset U \cap U_1$ and $P|_{U_2}^{-1}\{b_1, ..., b_m, ...\} \subset U \cap U_2$. Since $U_1 \cap U_2 = \phi$, this contradicts injectiveness on U. So P is injective on boundary(U).

To see that boundary(U) is compact, notice that M, being a manifold, is metrizable. So the Bolzano-Weierstrass theorem applies. Let $\{z_1, ..., z_m, ...\} \subset$ boundary(U). $\{P(z_1), ..., P(z_m), ...\}$, which is an infinite set since P is injective on boundary(U), has a limit point $b \in$ boundary(B). There's a point $z \in$ boundary(U) with $P(z) = b$, by assumption. Let U_1 be an open set containing z that is mapped homeomorphically to a ball in $\mathbb{C}^n$ centered at b. There is an infinite set $\{z'_{i_1}, ...z'_{i_m}, ...\} \subset U_1$ such that $P(z'_{i_m}) = P(z_{i_m})$.

$P(z'_{i_m})$ is the limit of a sequence $\{b_1, ..., b_k, ...\} \subset P(U_1) \cap B$. So z'_{i_m} is the limit of

$\{P^{-1}(b_1), ..., P^{-1}(b_k), ...\} \subset U_1 \cap P^{-1}(B)$, which is contained in U, as was proved above. So $z'_{i_m} \in$ boundary(U). Since P is injective on boundary(U), $z'_{i_m} = z_{i_m}$, showing that z is a limit point of $\{z_1, ..., z_m, ...\}$. So boundary(U) is compact.

So now we can show that P is a homeomorphism to a larger ball containing B. Let $\{U_1, ..., U_m\}$ be a finite cover of boundary(U) with open sets U_i such that P maps $\overline{U_i}$ homeomorphically to a closed ball in $\mathbb{C}^n$, centered on boundary(B). There's a larger ball $B' \supset B$ centered at $P(Z)$ with $B' \subset B \bigcup_i P(U_i)$.

Suppose for $z_1, z_2 \in U \bigcup_i U_i$, $P(z_1) = P(z_2)$. $P(z_1) \notin B$ because $U_i \cap P^{-1}(B) \subset U$. $P(z_1) \notin$ boundary(B) because, as was shown earlier, $U_i \cap P^{-1}$ (boundary(B)) $\subset$ boundary(U), and P is injective on boundary(U).

So if P is not injective in a ball larger than B centered at $P(Z)$, then for some i, j there are infinitely many points $\{c_1, ..., c_m, ...\}$ in $P(U_i) \cap P(U_j)$ approaching boundary(B), for which $P|_{U_i}^{-1}(c_m) \neq P|_{U_j}^{-1}(c_m), \forall m$. Since the c_m's are contained in a compact subset of $\mathbb{C}^n$, they have an accumulation point c, which is on boundary(B). By passing to a subsequence if necessary, we can assume that $\{c_1, ..., c_m, ...\}$ approaches c. There is a unique $z \in$ boundary(U) such that $P(z) = c$. z has a neighborhood V that is homeomorphic to a ball centered at c. Since P is injective on $\overline{U_i}$ and $\overline{U_j}$, $P|_{U_i}^{-1}(c_1, ..., c_m, ...)$ and $P|_{U_j}^{-1}(c_1, ..., c_m, ...)$ converge to z; so for $m >$ some finite number, $P|_{U_i}^{-1}(c_m) \in U_i \cap V$ and $P|_{U_j}^{-1}(c_m) \in U_j \cap V$. But then P isn't injective in V, a contradiction. So P is injective in a ball concentric with B and larger than B. $\square$

The proof of the lemma also works if B is replaced by $\Delta_M(Z; R)$.

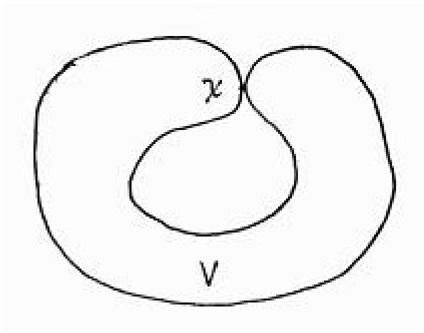
It's easy to see that $\{\epsilon \colon B_M(Z; \epsilon) \subseteq M\}$ is closed. So the lemma shows that $d_M(Z)$ really is the "distance to the boundary".

The "long lemming" can be extended somewhat:

Long Lemma+: Suppose U is an open neighborhood of $z \in M$, where (M, P) is a Riemann domain, and P is a homeomorphism from U to an open set V in $\mathbb{C}^n$, and for every point $b \in$ boundary(V), there's a point $z \in$ boundary(U) with $P(z) = b$. Suppose further that there's a homeomorphism f from an open set V_0 containing $\overline{V}$ into $\mathbb{C}^n$, such that $f(V)$ is the ball $B(0; 1)$.

Then P is injective on an open set U' containing $\overline{U}$.

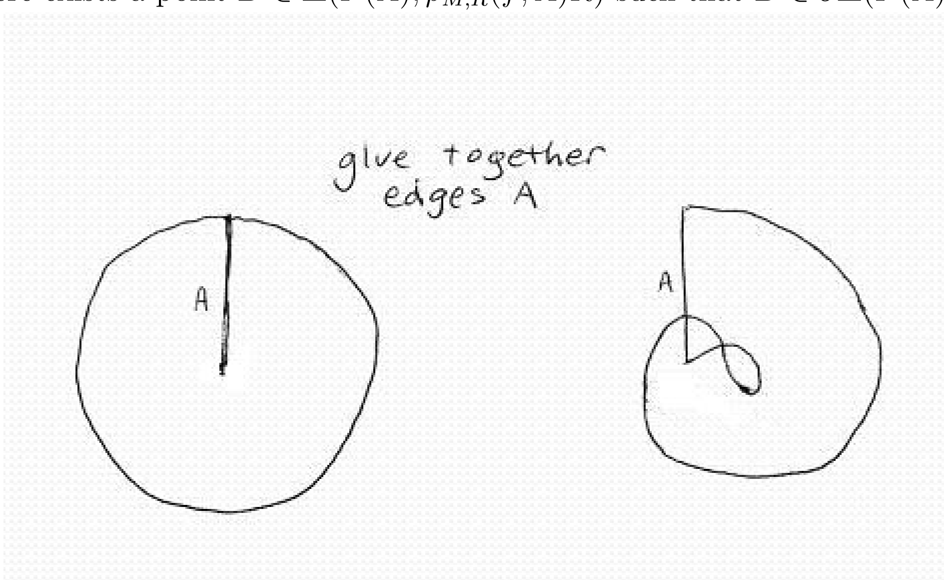
If there is no such homeomorphism from an open neighborhood of $\overline{V}$ to $\mathbb{C}^n$, then it may not be possible to extend P injectively. For example:



The point x might have two different preimages in $\overline{U}$, so P wouldn't be injective on an open set containing $\overline{U}$. V is homeomorphic to a ball, but the homeomorphism can't be extended to an open neighborhood of $\overline{V}$.

p. 76 Actually the spectrum of $\mathcal{O}_M$ is the set of *nonzero* continuous algebra homomorphisms $T : \mathcal{O}_M \to \mathbb{C}$.

p. 83, towards the end: "with the second inclusion necessarily a proper inclusion, and that there exists a point $B \in \Delta(P(A); \rho_{M,R}(f; A)R)$ such that $B \in \partial\Delta(P(A); \delta_{M,R}(A)R) \cap \partial P(U)$"



What if M looks like this?
As A approaches the center of the circle from the left, $\delta_{M,R}(A) \to 0$ so

$\Delta(P(A); \delta_{M,R}(A)R) \subset P(U)$, if $U = M$.

The proof of this theorem in Gunning and Rossi doesn't use this argument, and it seemed fine to me. It's Theorem G5 in that book.

p. 97, third line from the top

$Mf_\nu(a;\rho) \to MF_\nu(a;\rho)$

p. 99, proof of Theorem J6 "The constants α_ν and β_ν are monotonically decreasing as ν tends to ∞ ..."

This isn't necessarily true, but the proof works all the same. Since

$$\alpha_\nu = \frac{Mf_\nu(a;r_1) - Mf_\nu(a;r_2)}{\ln r_1 - \ln r_2} \text{ and } \beta_\nu = \frac{Mf_\nu(a;r_2)\,\ln r_1 - Mf_\nu(a;r_1)\,\ln r_2}{\ln r_1 - \ln r_2},$$

α_ν and β_ν aren't necessarily monotonically decreasing.

But $MF_\nu(a;r)$ is linear in ln r, so if either $\lim_\nu MF_\nu(a;r_1) = -\infty$ or $\lim_\nu MF_\nu(a;r_2) = -\infty$, then $\lim_\nu MF_\nu(a;r) = -\infty$ for almost all $r \in [r_1, r_2]$, which is a contradiction since $Mu(a;r) > -\infty$ for almost all $r \in [r_1, r_2]$.

p. 102 A function $u\colon \mathbb{C}^n \to R$ is <u>harmonic</u> if $\displaystyle\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} + \frac{\partial^2 u}{\partial y_j^2} = 0$.

p. 108, Lemma K10 - the second assertion is that if u is locally Lebesgue integrable, the functions u_ϵ converge to u in L^1 norm on any compact subset of D.

What the proof shows is that the $\{u_\epsilon\}$ converge to u in L^1 norm on an open set $U \subset D$ with compact closure in D.

But why not substitute an arbitrary compact set $K \subset D$ for $\overline{U}$ in the proof, and integrate over K instead of over U?

By the way, the boundary of an open set in $\mathbb{C}^n$ doesn't necessarily have measure 0, so integrating over $\overline{U}$ isn't equivalent to integrating over U. For example, let $U = \bigcup U_i$, where the U_i are balls of measure $\frac{1}{2^i}$, centered at the rational points in $\Delta(0; 10)$.

Same comment applies to the third assertion of Lemma K10.

p. 110, last paragraph: "it follows from Fatou's lemma in measure theory that ..."

I'm not sure if Fatou's lemma can be applied here, but it's not necessary. Because u is upper semicontinuous, for a given $\delta > 0$ you can find ϵ such that $\sup\{u(Z+\epsilon W), |W| < 1\} < u(Z)+\delta$.

Then

$$u_\epsilon(Z) = \int_{\Delta(0;1)} u(Z+\epsilon W)\sigma(W)dV(W) < \int_{\Delta(0;1)} (u(Z)+\delta)\sigma(W)dV(W) = u(Z)+\delta.$$

Since δ was arbitrary, $\lim_{\epsilon \to 0} u_\epsilon \leq u$.

The reason Fatou's lemma may not apply is that we'd be taking the limit of monotonically decreasing functions $g_n(W) = \sup\{u(Z+\epsilon W), 0 \leq \epsilon < 1/n\}$, and it's not clear to me that g_n must be measurable.

p. 116 towards the end:

Should be "the pseudonorms $||f||_\nu = \int_{K_\nu} |f(Z)|dV(Z)$ define this topology", rather than $\int_{K_\nu} |f(Z)|^2 dV(Z)$.

$L^1_{loc}(R^n) \nsubseteq L^2_{loc}(R^n)$, although $L^2_{loc}(R^n) \subset L^1_{loc}(R^n)$ because of the Schwarz inequality.

p. 117 Corollary K17. The conclusion of the proof should be "u is necessarily equal to a plurisubharmonic function almost everywhere in D."

p. 119 The proof of Theorem L2 uses the fact that the roots of a polynomial are continuous functions of its coefficients, in the statements about the eigenvalues of Lv and Lu'_A.

Click here for a proof of this fact.

This means that $\mathbb{C}^n/S_n$, the quotient space of $\mathbb{C}^n$ by permutations on the coordinates, is homeomorphic to $\mathbb{C}^n$ via the map between roots of polynomials of degree n and their coefficients (!)

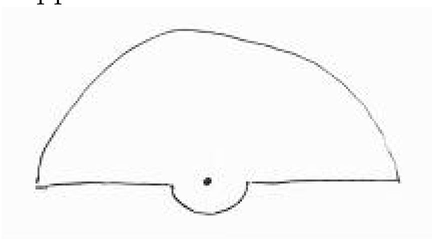
p. 130, first paragraph: "it follows from the assumption that D is pseudoconvex in the sense of Hartogs that I is a closed subset of $[0, 1]$."

It doesn't follow directly. Slightly altered proof: An open subset of $[0, 1]$ is a union of disjoint open intervals intersected with $[0, 1]$. If I isn't the whole of $[0, 1]$, a connected component of I is an open or half-open interval, e.g. $(a, 1]$. But, because D is pseudoconvex in the sense of Hartogs, I also contains a, which is a contradiction.

p. 130, after the proof of Theorem M3: “d_D ... can be replaced by any function d of the form ... where E is any open neighborhood of the origin in $\mathbb{C}^n$ with the property that $tE \subseteq E$ whenever $t \in [0, 1]$.”

This is not true. The continuity of d_D depends on the fact that if Z' is close to Z, then a ball around Z contains a slightly smaller ball around Z'.

Suppose the set E looks like this:



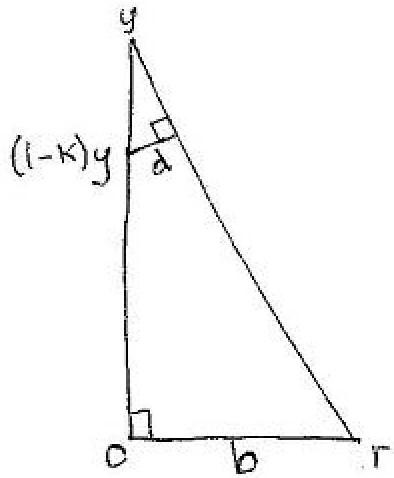
The point at the center is the origin. If Z' is displaced slightly downwards from Z, an E-shaped neighborhood around Z doesn't contain a slightly smaller E-shaped neighborhood around Z', so the distance function d_E defined by E is not continuous.

E can be described as “starshaped around 0”, meaning that $tE \subseteq E$ for $t \in [0, 1]$. Similarly, E would be *starshaped around* w if $t(E - w) + w \subseteq E$ for $t \in [0, 1]$.

Continuity condition: If E is an open neighborhood of the origin in $\mathbb{C}^n$, and there is an open neighborhood U of the origin such that for any $z \in U$, E is starshaped around z, then the distance function d_E defined on a domain $D \subseteq \mathbb{C}^n$ by $d_E(z) = \sup\{k : kE + z \subseteq D\}$ is continuous.

Proof: We want to show that if $z', z \in D$, $|z' - z| < \delta$, then $|d_E(z') - d_E(z)| < \epsilon$, where δ may depend on z. It's enough to show that $(d_E(z) - \epsilon)E + r \subseteq d_E(z)E$ for $|r| \leq \delta$ if δ is small enough; this means that $d_E(z') \geq d_E(z) - \epsilon$ if $|z' - z| \leq \delta$. By symmetry, this would mean that δ can also be found small enough so that $d_E(z') \leq d_E(z) + \epsilon$.

U contains a ball of radius $2b > 0$ and E contains a ball of radius $2B > 0$, both centered at the origin. B can be taken to be $\geq b$. Let $y \in E$, and $r \in \mathbb{C}^n$, $|r| = b$ and r perpendicular to y, as vectors in $\mathbb{R}^{2n}$. Then the right angle triangle spanned by $0, y, r$ is contained in E (see diagram).



So a circle of radius $d = \dfrac{kb}{\sqrt{1 + \dfrac{b^2}{|y|^2}}}$ centered at $(1-k)y$ is contained in E.

Since this is true for all r with $|r| = b$ and perpendicular to y, a ball of radius d centered at $(1-k)y$ is contained in E.

If $|y| < B$, a ball of radius B centered at $(1-k)y$ is contained in E. Since $B \geq b$, this means that for all $y \in E$, a ball of radius $\dfrac{kb}{\sqrt{1 + \dfrac{b^2}{B^2}}}$ centered at $(1-k)y$ is contained in E.

So, to find δ such that $d_E(z') \geq d_E(z) - \epsilon$ if $|z' - z| < \delta$, set

$$k = \frac{\epsilon}{d_E(z)}, \text{ so } \delta = \frac{\epsilon b}{\sqrt{1 + \dfrac{b^2}{B^2}}}.$$

d_E is actually uniformly continuous! □

The continuity condition is probably necessary as well as sufficient.

Also, E must be bounded if it is to generate a distance function d_E that can be substituted for d_D in Theorem M3. Otherwise d_E might be 0 for $z \in D$; then $-\ln(d_E)$ wouldn't be plurisubharmonic.

If E is bounded and it satisfies the continuity condition, d_E can be substituted for d_D in Theorem M3.

p. 131, paragraph just before Theorem M6:

"a holomorphically convex subset $D \subseteq \mathbb{C}^n$ does not admit an exhaustion function of the form $|f|$ where f is holomorphic in D. That is a well-known observation in the case $n = 1$."

$D = \mathbb{C}$ is a counterexample in the case $n = 1$. Since any open set in $\mathbb{C}$ is holomorphically convex, D is. The sets $|z| \leq r$ are compact, so $|z|$ is an exhaustion function.

It is true if $D \subset \mathbb{C}$ is bounded though. Suppose f is analytic on D and $|f|$ is an exhaustion function. Since the zeroes of f are contained in a compact subset of D, there are only finitely many of them. So f can be divided by a polynomial $p(z)$, resulting in a function f' that is nonzero in D. Since $p(z)$ is bounded on D, $|f'|$ is also an exhaustion function on D. But then by the maximum modulus theorem, $1/f'$ is constant on the connected component of D where f' attains its minimum, so f' can't be an exhaustion function.

It's also true in $\mathbb{C}^n, n > 1$. In that case, take K to be $\{z \in D : f(z) = 0\}$. K is a thin subset of D (see Defn. D1). By Corollary D3, $D - K$ is connected. So Theorem E6 can be applied.

p. 131, proof of Theorem M6: "$d_D(Z) = \lim_\nu d_{D_\nu}(Z)$"

It's clear that $d_D(Z) \geq \lim_\nu d_{D_\nu}(Z)$. Suppose $d_D(Z) > \lim_\nu d_{D_\nu}(Z)$. Then there's a closed ball $\overline{B}(Z;r)$ of finite radius which has non-empty intersection with any $D_\nu^\complement$, but $\overline{B}(Z;r) \cap \bigcap_\nu D_\nu^\complement$ is empty. Since $\overline{B}(Z;r)$ is compact, this can't happen.

p. 133 In the proof of Theorem M9, there's a statement that if B is an open connected subset of $\mathbb{R}^n$, $-\ln d_B$ is a convex function in B iff B is convex. This is true, and here's a

Proof: If B is convex, and the line segment $\overline{x_1x_2} \subset B$, then

$$d_B(\lambda_1 x_1 + \lambda_2 x_2) \geq \lambda_1 d_B(x_1) + \lambda_2 d_B(x_2), \text{ for } \lambda_1, \lambda_2 > 0 \text{ and } \lambda_1 + \lambda_2 = 1.$$

$$\text{So } \ln\,(d_B(\lambda_1 x_1 + \lambda_2 x_2)) \geq \ln(\lambda_1 d_B(x_1) + \lambda_2 d_B(x_2)) \geq \lambda_1 \ln(d_B(x_1)) + \lambda_2 \ln(d_B(x_2)),$$

so $-\ln\, d_B$ is convex.

Now suppose B isn't convex. Then there are points $x_1, x_2 \in B$ with $\overline{x_1x_2}$ not contained in B. But there's a path P in B between x_1 and x_2. P can be covered by a finite number of balls centered at points in P and contained in B. x_1 can be connected to x_2 via the centers of these balls, by a path consisting of line segments contained in B.

So, let P' be a path in B using the minimal number of line segments to connect x_1 and x_2. Let $\overline{a_1a_2}$ and $\overline{a_2a_3}$ be adjacent line segments in P', and $a_t = (1-t)a_2 + ta_3$, for t between 0 and 1. Let $t_0 = \inf\{t : \overline{a_1a_t} \not\subseteq B\}$. Then $\overline{a_1a_{t_0}}$ is contained in $\overline{B}$ but not in B.

For some t slightly less than t_0, the path $\overline{a_1a_t}$ satisfies:

$\max(-\ln\, d_B)$ on $\overline{a_1a_t}$ $> \max(-\ln\, d_B(a_1), -\ln\, d_B(a_{t_0})) + 1$, and

$-\ln d_B(a_t) < -\ln d_B(a_{t_0}) + 1$.

Then on $\overline{a_1 a_t}$, $-\ln d_B$ isn't convex.

p.138 "If every plurisubharmonic function in D extends to a pseudoconvex set $E \supset D$, then since D is preserved by arbitrary translations in the imaginary direction, it is evident that ... E must also be a tube domain."

This is only clear if we know that pseudoconvex domains are domains of holomorphy, which hasn't been proved yet.

Suppose we tried to prove it, only knowing that E is pseudoconvex. Assuming E is properly contained in convex hull(D), from Theorem M3 the function $u = -\ln d_E$ is plurisubharmonic in E and can't be extended to ch(D). But this doesn't serve to derive a contradiction. Using the invariance of D under imaginary translations, for $s \in \mathbb{R}^n$ we can define a function u_s which is plurisubharmonic on $E + is$ and is equal to u on D. But extensions of plurisubharmonic functions aren't necessarily unique, so we can't conclude from this that u must have a plurisubharmonic extension to ch(D).

If we know that a pseudoconvex open set is a domain of holomorphy, then it's easy to see that E must be ch(D). If $E \subsetneq$ ch(D), then there a point $z \in$ boundary(E)$\cap$ch(D). If A_ν is a discrete sequence of distinct points in E approaching z, by theorem G7 there's a function $f \in \mathcal{O}_E$ such that $\limsup_\nu |f(A_\nu)| = \infty$. So f can't be extended to ch(D) contrary to Theorem D12.

p. 140, proof of Theorem N3, (i) $\Rightarrow$ (ii) step:

G can be extended to $1 \times \overline{\Delta}(0;1)$ by defining $G(1,z)$ as an accumulation point of $G(t,z)$ as $t \to 1$ (Call the accumulation point p). How does it follow that G is continuous on $[0,1] \times \overline{\Delta}(0;1)$?

There's a neighborhood U of p that is mapped homeomorphically by P to a ball $B(P(p);r)$ in $\mathbb{C}^n$.

Since F is continuous on $[0,1] \times \overline{\Delta}(0;1)$, for $\epsilon > 0$ there is δ_ϵ such that if $||(t,z') - (1,z)|| < \delta_\epsilon$, $||F(t,z') - F(1,z)|| < \epsilon$. Here $[0,1] \times \overline{\Delta}(0;1)$ is given a metric as a subset of $\mathbb{R}^3$. Clearly, $P(p) = F(1,z)$.

So $G(B((1,z);\delta_r) \cap [0,1) \times \overline{\Delta}(0;1)) \subset P^{-1}(B(P(p);r))$. But, from the Long Lemming (see my comments on pg. 73), U is the whole of a connected component of $P^{-1}(B(P(p);r))$. Since $B((1,z);\delta_r) \cap [0,1) \times \overline{\Delta}(0;1)$ is connected, $G(B((1,z);\delta_r) \cap [0,1) \times \overline{\Delta}(0;1)) \subset U$ and p is the only accumulation point of $G(t,z')$ as $(t,z') \to (1,z)$. So if $G(1,z)$ is defined to be p, G is continuous on $[0,1] \times \overline{\Delta}(0;1)$.

p. 141, proof of Theorem N3, (ii) $\Rightarrow$ (iii) step:

First, the notation can be simplified a lot by observing that there's a homeomorphism of $\mathbb{C}^n$ that sends $A+Bz$ to $z(1,0,...,0)$. So without loss of generality one can assume that $P(H(z)) = (z,0,...,0)$. In what follows I'll suppress the other dimensions of $\mathbb{C}^n$, and simply write $P(H(z)) = z$.

The proof requires extending G_W from $0 \times \overline{\Delta}(0;1)$ to $[0,\delta) \times \overline{\Delta}(0;1)$ and $[0,1] \times \partial\Delta(0;1)$. Since G_W must equal $P^{-1}F_W$, G_W is well-defined and continuous if P maps a connected subset of M containing $H(\overline{\Delta}(0;1))$ injectively onto the range of F_W.

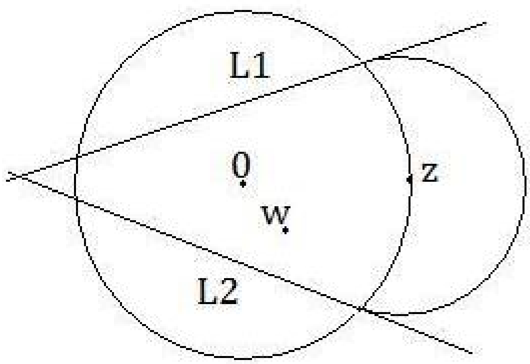
G_W can be extended to $[0,1] \times \partial\Delta(0;1)$ via the Long Lemma+ (see my comments on p. 73). Let T be the sup of all t such that P maps a connected subset of M containing $H(\overline{\Delta}(0;1))$ injectively onto

$$U_t = \Delta(0;1) \cup \bigcup_{z \in \partial\Delta(0;1)} B(z; t|e^{-p(z)}|),$$

where $B(z;t|e^{-p(z)}|)$ is a ball in the one-dimensional complex plane.

The aim is to show that $T \geq 1$, since then G_W can be extended to $[0,1] \times \partial\Delta(0;1)$. $T > 0$ since $P \circ H$ is injective on an open neighborhood of $\overline{\Delta}(0;1)$.

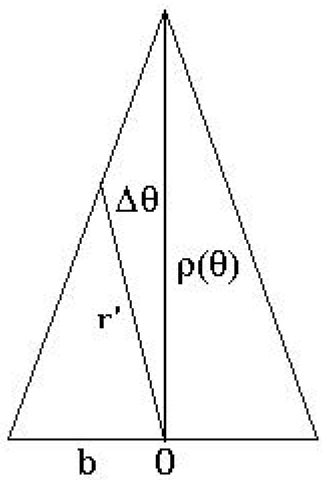
For a single point z, $U_{t,z} = \Delta(0;1) \cup B(z;t|e^{-p(z)}|)$ is "smoothly starshaped", meaning that there is open neighborhood V of 0 such that for $w \in V$ and $s \in [0,1]$, $sU_{t,z} + (1-s)w \subseteq U_{t,z}$. See diagram below.



If w is between lines L1 and L2 in $B(0;1)$, $U_{t,z}$ is starshaped around w.

Since $t|e^{-p(z)}|$ has a nonzero minimum on $\partial\Delta(0;1)$, U_t is also smoothly starshaped.

Since U_t is smoothly starshaped, the function $\rho(\theta) = \sup\{r : re^{i\theta} \in U_t\}$ is continuous, as you can see from the diagram below.



U_t is starshaped with respect to a ball $B(0;b)$, so, since the line from 0 to the apex of the triangle is in U_t, the whole triangle is in U_t. As $\Delta\theta \to 0, r'$ approaches $\rho(\theta)$. Since $\rho(\theta + \Delta\theta) \geq r', \rho(\theta)$ is continuous.

So f_t defined by $f_t(z) = z/\rho(\theta(z))$ for $z \neq 0$ and $f_t(0) = 0$ is a homeomorphism of $\mathbb{C}$ which maps U_t to $B(0;1)$.

Since (M, P) is a Riemann domain, $M_{\mathbb{C}} = P^{-1}(\mathbb{C} \times 0 \times ... \times 0)$ is also a Riemann domain, with projection map P. Let $W_T \subset M_{\mathbb{C}}$ be the connected open set containing $H(\overline{\Delta}(0;1))$ that is mapped injectively to U_T by P.

Suppose that $T < 1$. Since $|e^{-p(z)}|$ is bounded away from 0 for $z \in \partial\Delta(0;1), \overline{U}_T \subset U_t$ for $T < t$, so $\overline{U}_T \subset \bigcap\limits_{t>T} U_t$.

Let b be a point in boundary(U_T). $d(b,c)/e^{-p(c)}$ is a continuous function of $c \in \partial\Delta(0;1)$, so it attains its infimum. The infimum must be T since otherwise $b \notin U_t$ for some $t > T$. So, $b \in \overline{B}(c, T|e^{-p(c)}|)$ for some $c \in \partial\Delta(0;1)$, and there's a neighborhood Y of $H(c)$ in $M_{\mathbb{C}}$ that's mapped injectively to $B(c, |e^{-p(c)}|)$, and $z \in Y$ such that $P(z) = b$. The subset of W_T that is mapped by P to $B(c, T|e^{-p(c)}|)$ and the subset of Y that is mapped to $B(c, T|e^{-p(c)}|)$ are the same, since they both contain $H(c)$. So $z \in$ boundary(W_T).

This means that for every point b' in boundary$(B(0;1))$, there's a point $z \in$ boundary(W_T) such that $P'(z) = b'$.

By the Long Lemma+, P is injective on an open set W' containing $\overline{W}_T$, and $P(W') \supset \overline{U}_T$.

Since $|e^{-p(z)}|$ is bounded for $z \in \partial\Delta(0;1)$, $U_{T'} \subset P(W')$ for some $T' > T$, which contradicts the maximality of T.

So $T \geq 1$, and G_W can be extended to $[0,1] \times \partial\Delta(0;1)$.

For a given $k \in (0,1)$, take δ_k to be the sup of δ's such that P maps an open neighborhood

of $H(\overline{\Delta}(0;1))$ in $M_{\mathbb{C}}$ injectively onto $\bigcup_{z\in\overline{\Delta}(0;1)} B(z;\delta k|e^{-p(z)}|) \bigcup \bigcup_{z\in\partial\Delta(0;1)} B(z;k|e^{-p(z)}|)$.
Since $|e^{-p(z)}|$ is bounded on $\overline{\Delta}(0;1)$, $\delta_k > 0$.

Suppose $\delta_k \leq 1$. If $|W| = k, G_W$ can be defined as a continuous function on $[0,\delta_k) \times \overline{\Delta}(0;1) \cup [0,1] \times \partial\Delta(0;1)$. Since M is pseudoconvex in the sense of Hartogs, G_W can be extended to a continuous map on $[0,\delta_k] \times \overline{\Delta}(0;1) \cup [0,1] \times \partial\Delta(0;1)$.

If $\delta_k < 1$, then

$$P\left(\bigcup_{|W|=k} G_W\left([0,\delta_k] \times \overline{\Delta}(0;1) \cup [0,1] \times \partial\Delta(0;1)\right)\right) = \bigcup_{z\in\overline{\Delta}(0;1)} \overline{B}(z;\delta_k k|e^{-p(z)}|) \bigcup \bigcup_{z\in\partial\Delta(0;1)} \overline{B}(z;k|e^{-p(z)}|).$$

This is a smoothly starshaped set, and closed by the same arguments as used above. P is injective onto the interior of $\bigcup_{z\in\overline{\Delta}(0;1)} \overline{B}(z;\delta_k k|e^{-p(z)}|)$, so by the Long Lemma+, P is injective onto an open neighborhood of $\bigcup_{z\in\overline{\Delta}(0;1)} \overline{B}(z;\delta_k k|e^{-p(z)}|)$.
Since $|e^{-p(z)}|$ is bounded on $\overline{\Delta}(0;1)$, this contradicts the maximality of δ_k.

So $\delta_k \geq 1$, which means that $d(H(z)) \geq k|e^{-p(z)}|$, $\forall z \in \Delta(0;1)$ and any $k \in (0,1)$. So $d(H(z)) \geq |e^{-p(z)}|$, $\forall z \in \Delta(0;1)$, which means that $-\ln(d_M)$ is plurisubharmonic in M.

p. 142, definition of $w(Z)$, near the top of page:

The set $\{Z : w(Z) \leq r\}$ isn't compact if $d_M(B) < r$, so w isn't an exhaustion function for M.

What's probably intended is to define $w(Z) = \max\ (-\ln\, d_M(Z), v(Z))$, where $v(Z)$ is the infimum of the length of continuously differentiable paths from Z to B. So defined, w really is a continuous exhaustion function for M.

The proof works, using the redefined w.

p. 142 equation (1) becomes

$$|v(Z_1) - v(Z_2)| \leq |P(Z_1) - P(Z_2)|$$

p. 142, middle of page

"it is only necessary to choose ϵ sufficiently small that $\Delta(0;\epsilon) \subset \Delta(0;1/\nu)$ or $\Delta(0;\epsilon) \subset \Delta(0;R/\nu)$"

Should be "$\overline{\Delta}(0;\epsilon) \subset B(0;1/\nu)$ or $\overline{\Delta}(0;\epsilon) \subset \Delta(0;R/\nu)$"

p. 142, last quarter of page:

"for any real number c the set $X = \{Z \in M : w(Z) < c\}$ has a compact closure in M, and therefore the set $\overline{X}_\epsilon = \bigcup_{Z\in X} \overline{\Delta}(Z;\epsilon)$ is a compact subset of M"

Should be "for any real number c the set $X = \{Z \in M_\nu : w(Z) < c\}$ has a compact closure in M, and therefore the set $\overline{X}_\epsilon = \bigcup_{Z\in \overline{X}} \overline{\Delta}(Z;\epsilon)$ is a compact subset of M"

$\overline{X}_\epsilon$ is compact because it's closed in M and it's contained in the compact set $\{Z \in M : w(Z) \leq \sup\limits_{Z\in\overline{X}_\epsilon} w(Z) < \infty\}$.

p. 142, near end of page:

"identifying an open neighborhood of the point $Z \in M$ with the polydisc ..."

Should be "$Z \in M_\nu$".

p. 143, middle of page

"$\delta^{-1}|w(Z+\delta E_j+\epsilon T) - w(Z+\epsilon R)| \leq \delta^{-1}||\delta E_j|| = 1$"

becomes with the new definition of w

"$\delta^{-1}|w(Z+\delta E_j+\epsilon T) - w(Z+\epsilon R)| \leq k^{-1}\delta^{-1}||\delta E_j|| = 1/k$,"

where k is the nonzero minimum value of d_M on the compact set $\overline{X}_\epsilon$.

This is because, for $Z_1, Z_2 \in \overline{X}_\epsilon$ and close enough to be in a neighborhood that's mapped homeomorphically to $\mathbb{C}^n$,

$|w(Z_1) - w(Z_2)| \leq \max\,(|\ln\, d_M(Z_1) - \ln\, d_M(Z_2)|, |v(Z_1) - v(Z_2)|)$, and

$$|\ln\, d_M(Z_1) - \ln\, d_M(Z_2)| = \frac{|d_M(Z_1) - d_M(Z_2)|}{d_M(Z')} \leq k^{-1}|d_M(Z_1) - d_M(Z_2)|,$$

for some $d_M(Z')$ between $d_M(Z_1)$ and $d_M(Z_2)$.

Since $|d_M(Z_1) - d_M(Z_2)| \leq |P(Z_1) - P(Z_2)|$, $|v(Z_1) - v(Z_2)| \leq |P(Z_1) - P(Z_2)|$ and $k < 1$, we get

$|w(Z_1) - w(Z_2)| \leq k^{-1}|P(Z_1) - P(Z_2)|$.

Because of this change, further down page 142 we have

$|Lw_\epsilon(Z;A)| \leq Ck^{-1}\epsilon^{-1}n^2||A||^2$ and

$v_\nu(Z) = w_\epsilon(Z) + 2Ck^{-1}\epsilon^{-1}n^2||P(Z)||^2$.

The rest of the proof is still valid with these changes.

p. 145, end of page: "for any integer ν the subset $M_\nu = \{z \in M : u(Z) < \nu\}$ also satisfies the hypothesis of this theorem and has a compact closure in M."

First, it should be "compact closure in N."

To see that M_ν satisfies the hypothesis of the theorem, first notice that the set $N_\nu = \{Z \in N : u(Z) < \nu\}$ is pseudoconvex, because it has a continuous plurisubharmonic exhaustion function

$$\phi(u(Z)) = \frac{1}{\frac{1}{u(Z)} - \frac{1}{\nu}}$$

$\phi \circ u$ is plurisubharmonic since ϕ is monotonically increasing and convex, and u is plurisubharmonic. (see Theorem K5(d))

Suppose $A \in \partial M_\nu = \partial(M \cap N_\nu) \subseteq \partial M \cup (M \cap \partial N_\nu)$. If $A \in \partial M$, by the hypothesis of the theorem there's a neighborhood U_A of A such that $U_A \cap M$ is pseudoconvex. By Theorem N5, $U_A \cap M_\nu = (U_A \cap M) \cap N_\nu$ is pseudoconvex.

If $A \in M \cap \partial N_\nu$, take U_A to be a ball centered at A and contained in M. U_A is pseudoconvex because any convex set is holomorphically convex, and holomorphically convex sets are pseudoconvex. So $U_A \cap N_\nu$ is the intersection of pseudoconvex sets, so it's pseudoconvex.

p. 146, top of page - Generalizing Theorem M11 to Riemann domains.

Condition (iii) in Theorem M11 reads "Whenever $K \subseteq D$ is compact, then the subset

$$\{A \in D : u(A) \leq \sup_{Z \in K} u(Z) \text{ for all plurisubharmonic functions } u \text{ in } D\}$$

is disjoint from an open neighborhood of ∂D"

What is an "open neighborhood of the boundary" for a Riemann domain?

On p. 101 of *From Holomorphic Functions to Complex manifolds* by Fritzsche and Grauert (viewable at books.google.com), an accessible boundary point of a Riemann domain M is defined as a sequence of points $\{Z_i\} \in M$ such that

1. $\{Z_i\}$ has no cluster point in M
2. The sequence $\{P(Z_i)\}$ has a limit point $z \in \mathbb{C}^n$
3. For every connected open neighborhood $V \ni z$ in $\mathbb{C}^n$, there is N such that if $n, m > N$, there's a path $\rho : [0,1] \to M$ between Z_n and Z_m such that $P \circ \rho([0,1]) \subset V$.

This suggests that for Riemann domains (iii) should be replaced by

(iii)$'$: Whenever $K \subseteq M$ is compact, then the subset

$$L = \{A \in D : u(A) \leq \sup_{Z \in K} u(Z) \text{ for all plurisubharmonic functions } u \text{ in } D\}$$

satisfies the condition that:

For a sequence of points $\{Z_i\} \subset L$, if

1. The sequence $\{P(Z_i)\}$ approaches a limit z in $\mathbb{C}^n$, and

2. For every $\epsilon > 0$, there is N such that for $n, m > N$, Z_n and Z_m can be connected by a path $\rho : [0,1] \to M$ such that $P \circ \rho([0,1]) \subset B(z; \epsilon)$,

then $\{Z_i\}$ has a cluster point in M.

Condition (iii)$'$ implies that the metric completion of L is contained in M, using the metric where $d(Z_1, Z_2)$ is the infimum of the lengths of continuously differentiable paths in M from Z_1 to Z_2.

The metric topology on M is the same as the topology it has as a Riemann domain.

If the projection map is injective on M, so M is homeomorphic to a domain in $\mathbb{C}^n$, then (iii) implies (iii)$'$.

However, (iii)$'$ doesn't imply (iii). Take $L = \{(x, \sin(1/x)), 0 < x \leq 1/2\}$, and let M be the union of balls $B((x,y); x^3), (x,y) \in L$.

L satisfies (iii)$'$, but M doesn't intersect the y-axis, so the interval $0 \times [-1,1] \subset \partial M$. But L isn't disjoint from an open neighborhood of $0 \times [-1,1]$, so it doesn't satisfy condition (iii). The points in $0 \times [-1,1]$ are not "accessible" boundary points of M, as defined by Fritzsche and Grauert. But the inaccessible boundary points don't matter in Theorem M11, since (iii)$'$ can be used to prove the theorem for Riemann domains.

Conditions (i) and (ii) in Theorem M11 can be used as is for Riemann domains.

For Riemann domains in general, (ii) implies (iii)$'$, since if L is compact, a sequence $\{Z_i\}$ in L has a cluster point.

(iii)$'$ $\Rightarrow$ (i) by an argument similar to that in Theorem M11. The essentially novel part is between hedges of $\looparrowright$.

Consider a continuous mapping $F : [0,1] \times \overline{\Delta}(0;1) \to \mathbb{C}^n$ such that F is holomorphic in $\Delta(0;1)$ for each fixed point of $[0,1]$ and $G : ([0,1) \times \overline{\Delta}(0;1)) \cup (1 \times \partial\Delta(0;1)) \to M$ is a continuous mapping such that $P \circ G = F$. If we show that G can be extended to a continuous map from $[0,1] \times \overline{\Delta}(0;1) \to M$, that will prove M is pseudoconvex.

The image $G([0,1] \times \partial\Delta(0;1)) = K$ is a compact subset of M, so condition (iii)$'$ applies to the set
$L = \{A \in M : u(A) \leq \sup\limits_{Z \in K} u(Z)$ for all plurisubharmonic functions u in $M\}$.

For any fixed point $t \in [0,1]$, any constant $\epsilon > 0$, and any plurisubharmonic function u in M, note that since $G(t \times \partial\Delta(0;1)) \subseteq K$ and u is upper semicontinuous in M, then
$u(A) < \sup\limits_{Z \in K} u(Z) + \epsilon$ for all points A in an open neighborhood of $G(t \times \partial\Delta(0;1))$ in M, and consequently $u(G(t,z)) < \sup\limits_{Z \in K} u(Z) + \epsilon$ for all points z in an open neighborhood of $\partial\Delta(0;1)$ in $\Delta(0;1)$. But $u(G(t,z))$ is actually a subharmonic function of $z \in \Delta(0;1)$, since $G(t,z)$ is a holomorphic function of z, so it follows from the maximum theorem for subharmonic functions

that $u(G(t,z)) < \sup_{Z\in K} u(Z) + \epsilon$ for all points $z \in \Delta(0;1)$. This is true for any value $\epsilon > 0$, and therefore $u(G(t,z)) \leq \sup_{Z\in K} u(Z)$ for all points $z \in \Delta(0;1)$. But the latter inequality is true for any plurisubharmonic function u, and therefore $G(t,z) \in L$ for all points $z \in \Delta(0;1)$. Thus, altogether $G([0,1) \times \overline{\Delta}(0;1)) \subseteq L$.

Suppose $\{(t_i, z_i)\} \subset [0,1) \times \overline{\Delta}(0;1)$ is a sequence approaching $(1,z)$, for some $z \in \overline{\Delta}(0;1)$.
Since $P \circ G(t_i, z_i) \to F(1,z)$, $\{G(t_i, z_i)\}$ satisfies 1. in condition (iii)$'$.

$\{G(t_i, z_i)\}$ also satisfies 2. in (iii)$'$: For $\epsilon > 0$, find δ such that if $|(t,z') - (1,z)| < \delta$, $|F(t,z') - F(1,z)| < \epsilon$. Then find N such that, for $n > N$, $|F(t_n, z_n) - F(1,z)| < \epsilon$. Then, for $n, m > N$ and $\lambda \in [0,1]$, $|P \circ G(\lambda(t_n, z_n) + (1-\lambda)(t_m, z_m)) - F(1,z)| < \epsilon$.

So, by (iii)$'$, $\{G(t_i, z_i)\}$ has a cluster point Z in M. Necessarily $P(Z) = F(1,z)$.

There's an open neighborhood $U \ni Z$ that is mapped homeomorphically by P to a ball $B(F(1,z);r)$ in $\mathbb{C}^n$. If s is small enough, $F(B((1,z);s)) \subseteq B(F(1,z);r)$, so $G(B((1,z);s)) \subseteq P^{-1}B(F(1,z);r)$. Since $G(B((1,z);s))$ is connected and U is the whole of a connected component of $P^{-1}B(F(1,z);r)$, $G(B(1,z);s)) \subseteq U$. So if $\{(t'_i, z'_i)\}$ is another Cauchy sequence in $[0,1) \times \overline{\Delta}(0;1)$ approaching $(1,z)$, $\{G(t'_i, z'_i)\}$ also approaches Z; so that $G(1,z)$ is well-defined as the limit of $G(t_i, z_i)$.

Since P is a local homeomorphism, the extended G is continuous on $[0,1] \times \overline{\Delta}(0;1)$, which proves the theorem.

p. 147, definition of the Riemann domain M:

In chapter O, it's generally assumed that M is σ-compact. By Theorem H3, if M is connected, it is second countable (that is, the topology of M has a countable basis) so it is σ-compact. The theorems and lemmas in this chapter can be proved for each connected component of M. So theorem 9, which is the the aim of the arguments in chapter O, is true for any pseudoconvex Riemann domain, σ-compact or not.

p. 148, definition of sign function at the top of the page:

The sign function is actually two different functions, and which is meant in the text depends on the context.

If I, J are complementary multi- indices, that is I is mapped to J by *, then the sign function is the Levi-Civita tensor in n dimensions:
$\text{sign}(IJ) = 1$ if $(i_1, ..., i_p, j_1, ... j_{n-p})$ is an even permutation of $(1, ..., n)$,

sign$(IJ) = -1$ if $(i_1, ..., i_p, j_1, ...j_{n-p})$ is an odd permutation of $(1, ..., n)$,
sign$(IJ) = 0$ if $(i_1, ..., i_p, j_1, ...j_{n-p})$ isn't a permutation of $(1, ..., n)$.
This is the definition that's used in equation (3).

If I, J aren't complementary multi-indices, then the sign function is as defined at the top of page 148. That's what's meant by sign(IM) and sign(JN) on p. 149.

p. 156, middle of page:
Should be

$$||\overline{\partial}\phi - \overline{\partial}\phi_\nu||_v^2 \le 2\sum_{I,J}^{*}\int_M e^{-v}(1-\rho_\nu)^2|(\overline{\partial}\phi)_{I,J}|^2 dV + 2\sum_{I,J}^{*}\int_M e^{-v}\left|\sum_K^{*}\sum_k \frac{\partial\rho_\nu}{\partial\overline{z}_k}\phi_{I,K}\ \mathrm{sign}(kKJ)\right|^2 dV,$$

so the expression on the RH side should be twice what's in the text. From (9),

$$(\overline{\partial}\phi - \overline{\partial}\phi_\nu)_{I,J} = (-1)^p \sum_K^{*}\sum_k \left(\frac{\partial\phi_{I,K}}{\partial\overline{z}_k}(1-\rho_\nu) - \phi_{I,K}\frac{\partial\rho_\nu}{\partial\overline{z}_k}\right)\ \mathrm{sign}(kKJ)$$

and setting

$$z_1 = \sum_K^{*}\sum_k \frac{\partial\phi_{I,K}}{\partial\overline{z}_k}(1-\rho_\nu),$$

$$z_2 = \sum_K^{*}\sum_k \phi_{I,K}\frac{\partial\rho_\nu}{\partial\overline{z}_k},$$

since $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - z_1\overline{z}_2 - \overline{z}_1 z_2$ and $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + z_1\overline{z}_2 + \overline{z}_1 z_2$,
$|z_1\overline{z}_2 + \overline{z}_1 z_2| \le |z_1|^2 + |z_2|^2$.
So $|z_1 - z_2|^2 \le 2|z_1|^2 + 2|z_2|^2$, which gives the expression above for $||\overline{\partial}\phi - \overline{\partial}\phi_\nu||_v^2$.
This change doesn't cause any problems in the rest of the proof.

p. 156, lower half of page:
"The second term is bounded from above by

$$\sum_{I,K}^{*}\int_M e^{-v}|\phi_{I,K}|^2\left(\sum_k \left|\frac{\partial\rho_\nu}{\partial\overline{z}_k}\right|^2\right) dV\text{''}$$

should be

$$2qn\sum_{I,K}^{*}\int_M e^{-v}|\phi_{I,K}|^2\left(\sum_k\left|\frac{\partial\rho_\nu}{\partial\overline{z}_k}\right|^2\right)dV$$

since for complex numbers $z_1, ..., z_q$, $|z_1 + ... + z_q|^2 \leq q\,(|z_1|^2 + ... + |z_q|^2)$, and similarly for the summation over k.

This change doesn't cause any problems in the rest of the proof.

p. 157, equation for $||\mathfrak{D}_{wu}\phi - \mathfrak{D}_{wu}\phi_\nu||_w^2$ at the top of the page:

The RH side of the equation should be twice what it is in the text, since (as in my comments on p. 156), for complex numbers z_1 and z_2, $|z_1 + z_2|^2 \leq 2|z_1|^2 + 2|z_2|^2$. Also, as Ken König commented, the exponent $-w - u$ should be replaced by $w - u$. And the first summation over I, J should have an asterisk on the top.

This change doesn't cause any problems in the rest of the proof.

p. 157, bottom of page

"v is also strictly plurisubharmonic": If $\rho' = 0$ this isn't true. So ρ is actually assumed to be strictly monotonically increasing.

p. 158 The integrals

$$\sum_{I,J}^{*}\int_{M_{\nu+1}-\overline{M}_\nu}\cdots$$

should be

$$\sum_{I,J}^{*}\int_{M_{\nu+1}-M_\nu}\cdots$$

p. 160, first paragraph

"If $\theta \in \mathcal{E}_M^{p,q-1}$ (should be $q-1$, not q) satisfies condition (i) of the preceding theorem, ... $\overline{\partial}\theta = \phi$."

If θ satisfies condition (i), then $(\sigma, \overline{\partial}\theta - \phi)_u = 0, \forall \sigma \in \mathcal{E}^{p,q}_{cM}$. Since $\overline{\partial}\theta - \phi \in \mathcal{E}^{p,q}_M$, $\rho_\nu(\overline{\partial}\theta - \phi) \in \mathcal{E}^{p,q}_{cM}$, so $\int_M \rho_\nu |\overline{\partial}\theta - \phi|^2 = 0$. Since $\rho_\nu = 1$ on U_ν, this means that $\int_{U_\nu - U_{\nu-1}} |\overline{\partial}\theta - \phi|^2 = 0$. This is true for all ν, so $\int_M |\overline{\partial}\theta - \phi|^2 = 0$; so $\overline{\partial}\theta - \phi = 0$.

p.161, end of page

"consider first an arbitrary linear partial differential operator $D : \mathcal{E}^{p,q}_M \to \mathcal{E}^{p,q}_M$ ..."

Is $(D\phi)_{I,J} = D(\phi_{I,J} dz_I \wedge d\overline{z}_J)$? Or could D, say, map $\phi_{I,J} dz_I \wedge d\overline{z}_J$ to $\phi_{I,J} dz_{I'} \wedge d\overline{z}_{J'}$? It's not clear, but the set $\mathcal{W}^{p,q}_\nu$ defined on p. 162 is the same, whichever definition is chosen. For the operators $D : \mathcal{E}^{p,q}_M \to \mathcal{E}^{p,q}_M$ used in the text, $(D\phi)_{I,J} = D(\phi_{I,J} dz_I \wedge d\overline{z}_J)$.

p. 163, first line of proof of Lemma 8:

Should be

"choose an open neighborhood Δ of $\tilde{A}$ in M such that an open neighborhood of $\overline{\Delta}$ is mapped biholomorphically",

since the norms in the proof are over $\overline{\Delta}$.

p. 163, middle of page

"the differential forms $D\psi_\epsilon$ converge uniformly on Δ as ϵ tends to zero. That evidently implies that the forms ψ_ϵ converge uniformly on Δ to a C^∞ differential form ψ as ϵ tends to 0."

To prove this, it's enough to show that if D is a first-order operator, say $D = \partial/\partial x_i$, then $D\psi$ exists and is equal to $\lim_{\epsilon \to 0} D\psi_\epsilon$.

For arbitrary $z \in M$,

$$\int_z^{z+\Delta x_i} \lim_{\epsilon \to 0} \partial\psi_\epsilon/\partial x_i = \lim_{\epsilon \to 0} \int_z^{z+\Delta x_i} \partial\psi_\epsilon/\partial x_i = \lim_{\epsilon \to 0} \psi_\epsilon(z + \Delta x_i) - \psi_\epsilon(z) = \psi(z + \Delta x_i) - \psi(z).$$

$\lim_{\epsilon \to 0} \partial\psi_\epsilon/\partial x_i$ is continuous, being the limit of a uniformly convergent series of continuous functions. So $\partial\psi/\partial x_i$ exists and is equal to $\lim_{\epsilon \to 0} \partial\psi_\epsilon/\partial x_i$.

p. 164, equation (23)

Ken König got his edit wrong here - the equation is right as it is.

p. 165, end of proof of Theorem 9: "The set K was arbitrary, and since $\lim_{\epsilon \to 0} D'\theta_\epsilon = \theta^{D'}$..."

To see that $\theta^{D'}$ exists, find compact sets $K_\nu : K_\nu \subset$ interior$(K_{\nu+1})$ and $\bigcup_{\nu=1}^{\infty} K_\nu = M$. On each K_ν, $D'\theta_\epsilon$ converges to $\sigma_\nu \in \mathcal{W}_0^{p,q-1}$ in $L^2_{K_\nu}$-norm. Since $||D'\theta_\epsilon - \sigma_{\nu+1}||_{K_\nu} \leq ||D'\theta_\epsilon - \sigma_{\nu+1}||_{K_{\nu+1}}$, $\sigma_{\nu+1} = \sigma_\nu$ on K_ν almost everywhere. So $\sigma_{\nu+1}$ can be redefined to be equal to σ_ν on K_ν. Then σ can be defined on M by $\sigma(\tilde{Z}) = \sigma_\nu(\tilde{Z})$ if $\tilde{Z} \in K_\nu$.

Now let $\alpha \in \mathcal{E}^{p,q}_{cM}$, with compact support K. $K \subseteq K_\nu$ for some ν.

$(D'\theta_\epsilon, \alpha) = (\theta_\epsilon, D'^*\alpha)$, and $\lim_{\epsilon \to 0} (D'\theta_\epsilon, \alpha) = (\sigma, \alpha)$ by the Schwarz inequality, and $\lim_{\epsilon \to 0} (\theta_\epsilon, D'^*\alpha) = (\theta, D'^*\alpha)$, also by the Schwarz inequality. So $(\sigma, \alpha) = (\theta, D'^*\alpha)$, which shows that $\theta^{D'}$ can be defined equal to σ. Which really does conclude this long detailed proof! Go have a cup of coffee!

p. 170, Oka's counterexample:

"However, the integral $\int_{\gamma_0} d \log h_0$ is nonzero"

I don't see how "γ_0 is homotopic to a loop in a complex plane transverse to M", so I did an explicit calculation.

$h_0 = 0$ at $(t_1, t_2) = (\pi/3, 2\pi/3)$.

Let γ_0 be an infinitesimal loop around $(\pi/3, 2\pi/3)$, with displacement $(-\Delta t_1, -\Delta t_2)$ to $(\Delta t_1, -\Delta t_2)$ to $(\Delta t_1, \Delta t_2)$ to $(-\Delta t_1, \Delta t_2)$ to $(-\Delta t_1, -\Delta t_2)$.

At $(z_1, z_2) = (e^{i\pi/3}, e^{2i\pi/3})$, the gradient of h is in the same direction as the gradient of $z_1 - z_2 - 1$, so $\partial h/\partial z_1 = \alpha$ and $\partial h/\partial z_2 = -\alpha$, for some $\alpha \in \mathbb{C}$. So h_0 varies around the loop from

$\alpha(-i\Delta t_1 e^{i\pi/3} + i\Delta t_2 e^{2i\pi/3})$ to
$\alpha(i\Delta t_1 e^{i\pi/3} + i\Delta t_2 e^{2i\pi/3})$ to
$\alpha(i\Delta t_1 e^{i\pi/3} - i\Delta t_2 e^{2i\pi/3})$ to
$\alpha(-i\Delta t_1 e^{i\pi/3} - i\Delta t_2 e^{2i\pi/3})$ to
$\alpha(-i\Delta t_1 e^{i\pi/3} + i\Delta t_2 e^{2i\pi/3})$.

Because $\arg(e^{i\pi/3}) \neq \arg(e^{2i\pi/3})$, the values of h_0 trace a quadrilateral around 0.

p. 170, end of page "for any C^∞ function f on M there exists a C^∞ function g on M such that $f = \Delta g$."

There's a proof of this, at least if M is an open subset of $\mathbb{C}^n$, in Lars Hörmander's *Linear Partial Differential Operators*(1969).

Here's how it works. A domain $D \subseteq \mathbb{R}^n$ is called *P-convex* if to every compact subset $K \subset D$

there exists another compact subset $K' \subset D$ such that $\phi \in C_0^\infty(D)$ and support $P(-D)\phi \subset K \Rightarrow$ support $\phi \subset K'$. (Def. 3.5.1 in Hörmander's book)

In the above, P is a polynomial in $\mathbb{R}[x_1, ..., x_n]$, and $D = -i(\partial/\partial x_1, ..., \partial/\partial x_n)$, so $P(-D)$ is a linear partial differential operator. In the case of the Laplacian operator Δ, $P = x_1^2 + ... + x_n^2$.

If D is P-convex, the equation $P(D)g = f$ has a solution in $C^\infty(D)$ for every $f \in C^\infty(D)$. (Corollary 3.5.2)

A differential operator $P(D)$ of order m is called *elliptic* if its principal part $P_m(D)$, that is, the homogeneous part of $P(D)$ of order m exactly, satisfies the condition $P_m(\vec{x}) \neq 0$ when $\vec{x} \in \mathbb{R}^n \neq 0$ (Def 3.3.2). The Laplacian, then, is an elliptic operator.

Every open set $D \subseteq \mathbb{R}^n$ is P-convex iff P is elliptic. (Corollary 3.7.1). $\square$

Laura