the secret behind magic circles

What the disk with q+1 pointers is pointing to, are the points on a line in $P^2(F_q)$. There are q+1 points on each line. Each pair of points in $P^2(F_q)$ is on only one line, just as in Euclidean geometry. That's why each distance between points occurs exactly once as a cyclically successive sum of the distances between the pointers. There are the same number of lines as points in $P^2(F_q)$ - that's why you can rotate the disk and find all the lines. It'll become more clear if you realize that a line in $P^2(F_q)$ comes from a plane passing through the origin in F_q^3 and a point in $P^2(F_q)$ comes from a line passing through the origin in F_q^3. Lines and planes in F_q^3 are dual to each other, that's why there's the same number.

The distances between pointers on the "magic circle" can be found from the field structure on F_q^3. If F is a finite field, it's of order $q = p^a$ for some prime p. And F_q^3 has a field structure, if you think of it as the field of order p^{3a}. So the nonzero elements of F_q^3 are a cyclic group, that is, all the group elements are powers of one element. So you can then arrange all the nonzero elements in a circle, $x, x^2, x^3, ...x^{q(q+1)}$. So find which powers of x describe points that are on a line, and that's the magic circle! When you multiply by x, it rotates the pointers by one jot. Multiplying by x is a linear transformation on F_q^3 , so it takes planes to planes in F_q^3, i.e. lines to lines in $P^2(F_q)$. So for finite fields you can squash $P^2(F_q)$, which looks 2-dimensional, onto a 1-dimensional circle. The "magic circle" also defines a distance between the lines in the projective space.

The space of lines in RP^2, the real projective plane, is homeomorphic to RP^2. Similarly for CP^2, the complex projective plane. So you could kind of do the same thing, moving a line over RP^2, so that any two points are on one translated version of the line, and all the lines are covered. But you'd be moving the line in 2 dimensions, since RP^2 really is 2-dimensional. Maybe you could squash RP^2 onto a "magic circle", with one line being an uncountable set of pointers rotating around. Perhaps by taking some kind of limit of what happens for finite fields. But it would be very traumatic for the topological structure of RP^2, it would hash it all up. It couldn't be a continuous map because of homology.

Time to go back to reading the book!

Laura